Group Rings

1. Connell's Theorem

For the rest of the paper, we'll let K be a field.

Connell's theorem is the following: $K[G]$ is a prime ring if and only if G has no finite normal subgroup. We can prove the forward direction easily through the contrapositive:

Proof: Let N be a finite normal subgroup of G. Consider the element $a = \sum_{n \in N} n$.

Notice that $a^2 = |N|\ a$. Then, defining $b = |N|\ 1 - a$, we see that:

$$ab = a(|N|\ 1 - a) = a\ |N| - a^2 = |N|\ a - |N|\ a = 0$$

□

The reverse direction is much tricker. Let's state some definitions that we'll need for later in this proof:

- $[g]$ is the conjugacy class for g. i.e. all elements that are conjugate with g.
- $\Delta(G) := \{g \in G \mid [g] \text{ is finite}\}$, also called the finite conjugate center.
- $\Delta^+(G) := \{g \in \Delta(G) \mid |g| < \infty\}$

We'll prove the reverse direction by showing the following chain of implications:

G has no finite normal subgroup $\Leftrightarrow \Delta^+(G) = \{1\} \Rightarrow K[\Delta G]$ is prime $\Rightarrow K[G]$ is prime.

We'll do so by proving the implications in order.

For the first implication, we'll need a few lemmas.

> **Lemma 1.1** **Dietzmann's Lemma**
>
> $[G : Z(G)] < \infty \Rightarrow |G'| < \infty$

We leave the proof for later in the paper.

> **Lemma 1.2** $\Delta^+ G < G$
>
> $\Delta^+(G) < G$

Proof of $\Delta^+(G) < G$***:***

Let N be the subgroup generated by finitely many elements $x_1, ..., x_n \in \Delta^+(G)$ and by their conjugates. Notice that $N \trianglelefteq G$ as for any $n \in N$, we must have that it's conjugate is also in N be definition.

Now, we wish to show that $|N| < \infty$. If this is true, then notice that for any $x, y \in \Delta^+(G)$, if we let $N = \langle x, y \rangle$ and its conjugates, then $|N| < \infty \Rightarrow |xy| < \infty$, and thus, xy has to be a torsion element with finitely many conjugates. So, $xy \in \Delta^+(G)$.

All that's left then, is to show that $|N| < \infty$. We wish to apply Dietmann's Lemma. Notice that $Z(N) = \cap_{i=1}^m C_N(x_i)$. Indeed, x_i has finitely many conjugates, so there are only finitely elements in the set $N/C_N(x_i)$. But that's precisely what it means to have finite index in N so, $|C_N(x_i)| < \infty$. Thus, $[N : Z(N)] < \infty$. So, by Dietmann'z, we see that $|N'| < \infty$.

Finally, notice that N/N' is finitely generated, abelian, and is torsion. Thus, $|N/N'| < \infty$. So, $|N| = |N'|\ |N : N'|$ and thus, $|N| < \infty$ □

Proposition 1.3 **Implication One**

G has no finite normal subgroup $\Leftrightarrow \Delta^+(G) = \{1\}$.

Proof of Implication One:

For the forward direction, by contrapositive, let $\Delta^+(G) \neq \{1\}$. Thus, there exists some $n \in \Delta^+(G)$ such that $n \neq 1$. Consider $N = \langle n \rangle$ alongside its conjugates. As shown above, $|N| < \infty$, and $N \trianglelefteq G$, so N is a finite normal subgroup of G.

For the reverse direction, by contrapositive, let G have a finite normal subgroup, say N. As $|N| < \infty$, we can write it as a finitely generated subgroup, say $\langle x_1, ..., x_k \rangle$. Then, $N < \Delta^+(G)$. Thus, $\Delta^+(G) \neq \{1\}$. □

We can now prove the next proposition. Recall that a commutative domain must be a prime ring. Thus, it suffices to show that $K[\Delta G]$ is a commutative domain.

Proposition 1.4 **Implication Two**

$\Delta^+ G = \{1\} \Rightarrow K[\Delta G]$ is prime.

Proof of Implication Two:

Notice that $\Delta^+(G) = \{1\}$, and thus, $\Delta(G)$ must be torsion-free. We wish to show that $\Delta(G)$ is abelian.

Now, for any $x, y \in \Delta(G)$, consider the subgroup generated $N = \langle x, y \rangle$. As these elements have a finite conjugate class, we see that $[N : C_N(x)] < \infty$ due to the same reasoning as in the proof of Lemma 0.2. Then, $Z(N) = C_N(x) \cap C_N(y)$ is a finite index subgroup. Thus, $[N : Z(N)] < \infty \Rightarrow |N'| < \infty$ by Dietzmann's.

Notice that N' is characteristic in N. Since N is normal in $\Delta(G)$, we must have that N' is normal in $\Delta(G)$. But since $|N'| < \infty$, we must have that $N' < \Delta^=(G) \Rightarrow N' = \{1\}$. Thus, $xy = yx$, so $\Delta(G)$ is abelian.

Finally, $\Delta(G)$ being abelian implies that $K[\Delta(G)]$ is commutative as K is a field and the only thing prohibiting commutativity are elements in G. To see that $K[\Delta(G)]$ is a domain is hard. This follows from Kaplansky's conjecture for abelian groups. □

We now need to somehow deduce the following:

Proposition 1.5 **Implication Three**

$K[\Delta G]$ is prime $\Rightarrow K[G]$ is prime.

To do so, we'll define a restriction map from any $H < G$, and show that it behaves relatively nicely with $K[H]$ and $K[G]$. The first lemma is simple.

Lemma 1.6 **Ideals and Group Rings**

Let $\pi : K[G] \to K[H]$ be the restriction map. Let I be an ideal of $K[H]$. Define $\rho(I) := IK[G]$. Then, $I \subset \rho(\pi(I))$. In particular, $I = 0 \Leftrightarrow \pi(I) = 0$.

Proof of Ideals and Group Rings:

We can write $K[G]$ as a $K[H]$ vector space. In particular, as $G = \bigcup Hg_i$ with g_i as coset representatives. We don't need normality as we I is a right ideal. Thus, we can write $K[G] = \bigoplus K[H]g_i$.

Now, let $x \in I$. Then, we can write $x = \sum x_i g_i$ with $x_i \in K[H]$ by the above decomposition. Then, $x_i = \pi(xg_i^{-1}) \in \pi(I)$ since I is a right ideal. So, $x_i g_i \in \rho(\pi(I))$, so $x \in \rho(\pi(I))$ as desired.

To see that $\pi(I) = 0 \Leftrightarrow I = 0$, notice that if $x = 0$, then $\pi(x) = \pi(0) = 0$. For the other direction, notice that $I \subseteq \rho(\pi(I)) = \rho(0) = 0 \cdot K[G] = 0$, as desired. □

In particular, choosing $H = \Delta(G)$ will be sufficient to prove implication three. The main lemma at play is the following:

Lemma 1.7 **Restrictions and Group Rings**

Let $\pi : K[G] \to K[\Delta G]$ be the restriction map. Let I, J be ideals of $K[G]$. Then, $IJ = 0 \Rightarrow \pi(I)\pi(J) = 0$.

We leave the proof for later in the paper.

We can now prove implication three.

Proof of Implication Three:

Let I, J be ideals in $K[G]$. Then, by the lemma above, if $IJ = 0$, then $\pi(I)\pi(J) = 0$. Since $K[\Delta G]$ is prime, then either $\pi(I)$ or $\pi(J)$ are zero. But by our first lemma, $\pi(I) = 0 \Leftrightarrow I = 0$. Thus, either $I = 0$ or $J = 0$, as desired. □

And with that, Connell's Theorem is complete, bar a few lemmas. Let's get those sorted out.

2. Lemmas for Connell's Theorem

We'll start by proving Dietzmann's Lemma. We need a specific commutator identity and commutator closure:

Lemma 2.1 **Commutator Identity**

$y[x, y]^n y^{-1} = [yxy^{-1}, y]^n$

Proof of Commutator Identity: We'll proceed by induction. First, we do the $n = 1$ case:

$$\begin{aligned} y[x,y]^n y^{-1} &= y(xyx^{-1}y^{-1})y^{-1} \\ &= (yx)e(yx^{-1}y^{-1}y^{-1}) \\ &= (yx)(y^{-1}y)(yx^{-1}y^{-1}y^{-1}) \\ &= (yxy^{-1})(y)(yx^{-1}y^{-1})(y^{-1}) \\ &= (yxy^{-1})(y)(yxy^{-1})^{-1}(y^{-1}) \\ &= [yxy^{-1}, y] \end{aligned}$$

Now the inductive step:

$$\begin{aligned}
y[x,y]^{n+1}y^{-1} &= y[x,y][x,y]^n y^{-1} \\
&= (yx)e(yx^{-1}y^{-1})e([x,y]^n y^{-1}) \\
&= (yx)(y^{-1}y)(yx^{-1}y^{-1})(y^{-1}y)([x,y]^n y^{-1}) \\
&= (yx)(y^{-1})(y)(yx^{-1}y^{-1})(y^{-1})(y[x,y]^n y^{-1}) \\
&= \left[(yxy^{-1})(y)(yxy^{-1})^{-1}(y^{-1})\right][yxy^{-1},y]^n \\
&= [yxy^{-1},y][yxy^{-1},y]^n \\
&= [yxy^{-1},y]^{n+1}
\end{aligned}$$

□

Lemma 2.2 **Commutator Closure**

If g, h are commutators, then $g^{-1}hg$ is also a commutator.

Proof of Commutator Closure: We have the following:

$$\begin{aligned}
g^{-1}hg &= g^{-1}[x,y]g \\
&= g^{-1}(xyx^{-1}y^{-1})g \\
&= (g^{-1}xg)(g^{-1}yg)(g^{-1}x^{-1}g)(g^{-1}y^{-1}g) \\
&= (g^{-1}xg)(g^{-1}yg)(g^{-1}xg)^{-1}(g^{-1}yg)^{-1} \\
&= [g^{-1}xg, g^{-1}yg]
\end{aligned}$$

□

We'll prove Dietzmann's Lemma in three main steps.

1. Show that G' is finitely generated.
2. Show that a commutator of large enough order can be reduced by using the identity above.
3. Show that every element in G' is a product of finitely many commutators.

By combining these three, we'll get our result.

Proof of Dietzmann's Lemma:

We can write G as follows:

$$G = \sqcup_{i=1}^{m} Z(G)h_i$$

We start by showing that G' is finitely generated. Notice that for $x, y \in G$, we can write $x = z_x h_x, y = z_y h_y$ for $z_x, z_y \in Z(G)$ and h_x, h_y as coset representatives. Then:

$$[xy] = xyx^{-1}y^{-1} = z_x h_x z_y h_y h_x^{-1} z_x^{-1} h_y^{-1} z_y^{-1} = h_x h_y h_x^{-1} h_y^{-1} = \left[h_x h_y\right]$$

Thus, there are m^2 many commutators as there are m choices for h_x and h_y. So G' is finitely generated. But we need to show some sort of order relation to reduce this to a finite set.

Now, we'll show that we can reduce a large power of a commutator to a slightly smaller power. In particular, we'll show the following for all $x, y \in G$:

$$[x,y]^{m+1} = [x,y^2][yxy^{-1},y]^{m-1}$$

Notice that $h^m \in Z(G)$ for every coset representative h as $h \in G \,/\, Z(G)$ and $|G \,/\, Z(G)| = m$. Thus, this extends to all $x \in G$, and in particular, $[x,y]^m$ is central. Now, we have:

$$\begin{aligned}
[x,y]^{m+1} &= [x,y]^m xyx^{-1}y^{-1} \\
&= xyx^{-1}[x,y]^m y^{-1} \\
&= xyx^{-1}([x,y][x,y]^{m-1})y^{-1} \\
&= (xyx^{-1}xyx^{-1}y^{-1})([x,y]^{m-1}y^{-1}) \\
&= (xyyx^{-1}y^{-1})e([x,y]^{m-1}y^{-1}) \\
&= (xy^2x^{-1}y^{-1})(y^{-1}y)([x,y]^{m-1}y^{-1}) \\
&= (xy^2x^{-1}y^{-1}y^{-1})(y[x,y]^{m-1}y^{-1}) \\
&= (xy^2x^{-1}y^{-2})(y[x,y]^{m-1}y^{-1}) \\
&= [x,y^2](y[x,y]^{m-1}y^{-1}) \\
&= [x,y^2][yxy^{-1},y]^{m-1} && \text{By Commutator Identity}
\end{aligned}$$

Now, all that's left is to show that an element in G' is a product of finitely many commutators. In particular, we claim that $g \in G'$ is a product of at most m^3 many commutators.

Assume not. So, the product is of length $r > m^3$. By Pigeonhole principle, there exists some commutator C such that C appears $m+1$ times in this product, since there are only m^2 many commutators.

We'll move C all the way to the left repeatedly. To do so, define ${g_i}' = C^{-1}g_iC$. This is still a commutator by commutator closure. Then:

$$\begin{aligned}
g &= g_1 \cdots g_i C g_{i+1} \cdots g_r \\
&= C(C^{-1}g_1C)(C^{-1}g_2C)\cdots(C^{-1}g_iC)g_{i+1}\cdots g_r \\
&= C{g_1}'\cdots{g_i}'g_{i+1}\cdots g_r
\end{aligned}$$

Do this for each occurance of C to get:

$$C^{m+1}g_1\cdots g_r$$

for possibly different g_i than in our original expansion. Then, notice that we can write C^{m+1} as a product of m commutators. Thus, we reduced r by 1. We can do this until $r = m^3$.

Indeed, we are done! Every element in G' is a product of at least m^3 commutators, and there are m^2 many commutators to choose from in each position. Thus, there are at most $(m^2)^{m^3}$ commutators, or a finite amount. Thus, $|G'| < \infty$.

□

All that's left now is Restrictions and Group Rings, which we'll restate before proving it. Before doing so, we'll need another lemma:

Lemma 2.3 **Neumann's Lemma**

Let $H_1, ..., H_m < G$ so that $G = \bigcup_{i=1}^{m}\bigcup_{j=1}^{n} H_i a_{ij}$. Then, at least one of H_i has finite index in G.

Proof of Neumann's Lemma:

We'll induct on m.

If $m = 1$, we are done as there are finitely many coset representatives.

Otherwise, let $m \geq 2$ so that H_1 has infinite index. Thus, we can find some coset, say $H_1 b$ not in $\cup_{j=1} H_1 a_{1j}$. But since it must exist in G, it must be the case that:

$$H_1 b \subseteq \bigcup_{i=2}^{m} \bigcup_{j=1}^{n} H_i a_{ij}$$

Multiplying by $b^{-1}a_{1k}$ for arbitrary $1 \leq k \leq n$, we get:

$$H_1 a_{1k} \subseteq \bigcup_{i=2}^{m} \bigcup_{j=1}^{n} H_i a_{ij} b^{-1} a_{1k}$$

The upshot is that we can now write $H_1 a_{1k}$ in terms of $m-1$ many cosets. So, we get:

$$\begin{aligned}
G &= \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} H_i a_{ij} \\
&= (H_1 a_{11} \cup \cdots \cup H_1 a_{1n}) \cup \left(\bigcup_{i=2}^{m} \bigcup_{j=1}^{n} H_i a_{ij} \right) \\
&\subseteq \left(\left(\bigcup_{i=2}^{m} \bigcup_{j=1}^{n} H_i a_{ij} b^{-1} a_{11} \right) \cup \ldots \cup \left(\bigcup_{i=2}^{m} \bigcup_{j=1}^{n} H_i a_{ij} b^{-1} a_{1n} \right) \right) \cup \left(\bigcup_{i=2}^{m} \bigcup_{j=1}^{n} H_i a_{ij} \right) \\
&= \left(\bigcup_{i=2}^{m} \bigcup_{j=1}^{n} \bigcup_{k=1}^{n} H_i a_{ij} b^{-1} a_{1k} \right) \cup \left(\bigcup_{i=2}^{m} \bigcup_{j=1}^{n} H_i a_{ij} \right)
\end{aligned}$$

Notice that in order to collapse this into one double union, we need to increase n so that all the coset representatives are included. Then, we are done by our inductive hypothesis, since one of H_i for $2 \leq i \leq m$ must have finite index in G. □

Now we're ready to prove the original lemma.

Lemma 2.4 **Restrictions and Group Rings**

Let $\pi : K[G] \to K[\Delta G]$ be the restriction map. Let I, J be ideals of $K[G]$. Then, $IJ = 0 \Rightarrow \pi(I)\pi(J) = 0$.

Proof of Restrictions and Group Rings:

Let $x \in I$ and $y \in J$. We want to show that $\pi(x)\pi(y) = 0$. As π is a projection, it is a bimodule map. Thus, $[\pi(x)]\pi(y) = \pi([\pi(x)]y)$. Indeed, $\pi(x)\pi(y)$ are formal sums of elements in $\Delta(G)$ as $\Delta(G) < G$, while $\pi(\pi(x)y)$ restricts to formal sums in $\Delta(G)$, and if $g_y \notin H$, then $hg_y \notin H$ for any $h \in H$. So, it suffices to show that $\pi(x)y = 0$.

Write $x = x_0 + x_1$ so that x_0 is the component from $\Delta(G)$ and x_1 is the component not from $\Delta(G)$. Thus:

$$x_0 = a_1 u_1 + \cdots + a_r u_r \text{ such that } u_i \in \Delta(G)$$
$$x_1 = b_1 v_1 + \cdots + b_s v_s \text{ such that } v_j \in \Delta(G)^C$$

Indeed, if we show that $x_0 y = 0$, we are done, since $\pi(x) = x_0$. We'll assume for the sake of contradiction that $x_0 y \neq 0$. Since $xy = 0$, we'll get a contradiction when we derive $x_1 = 0$. We'll use Neumann's Lemma to show that some v_j has finite centralizer and contradicting that $v_j \in \Delta(G)^C$. So, x_1 has to be zero, and we're done.

In particular, define $C \coloneqq \bigcap C(u_i)$. Then, as $[G : C] < \infty$, we can write $G = \bigcup_{k=1}^{n} C h_k$. But if we can write $C = \bigcup_{i=1}^{s} \bigcup_{j=1}^{m} C(v_i) g_{ij}$, we can definitely write G as a finite union of cosets of $C(v_i)$ as follows:

$$G = \bigcup_{k=1}^{n} \left[\bigcup_{i=1}^{s} \bigcup_{j=1}^{m} C(v_i) g_{ij} \right] h_k = \bigcup_{i=1}^{s} \bigcup_{l=1}^{t} C(v_i) k_l$$

And thus, one of $C(v_i)$ must be finite, a contradiction.

Notice that showing $C \subseteq \bigcup_{i=1}^{s} \bigcup_{j=1}^{m} C(v_i) g_{ij}$ is sufficient since all we need to do is cover G.

Assume for the sake of contradiction that $x_0 y \neq 0$. Choose $g \in G$ so that g appears in $x_0 y$ with a nonzero coefficient. Now, let $y = c_1 w_1 + \cdots + c_t w_t$ with $w_i \in G$. Then, define g_{ij} as follows:

$$g_{ij} \coloneqq \begin{cases} g_{ij} v_i g_{ij}^{-1} = g w_j^{-1} \text{ if } v_i \text{ is conjugate to } g w_j^{-1} \\ 1 \qquad\qquad\qquad\qquad \text{otherwise} \end{cases}$$

In other words, we choose g_{ij} to be a specific conjugate if possible, and 1 otherwise. We'll show that if $h \in C$, then $g w_j^{-1}$ must have a conjugate. So, we won't need to worry about the 'otherwise' case.

Now, we can start showing our inclusion. Let $h \in C$. We want to show it exists in some coset $C(v_i) g_{ij}$ for some suitable i, j. To do so, we'll show $h = g_{ij}$. Thus, $e = h g_{ij}^{-1} \in C(v_i)$. Thus, $h \in C(v_i) g_{ij}$.

Notice that $hxh^{-1} y \in IJ$ as $hxh^{-1} \in I$. So, $hxh^{-1} y = 0$. Also notice that h commutes with x_0 as $h \in \bigcap C(u_i)$. Also, $g \in x_0 y$. Expanding yields:

$$\begin{aligned} hxh^{-1} y &= 0 \\ h(x_0 + x_1) h^{-1} y &= 0 \\ (h x_0 h^{-1} y) + (h x_1 h^{-1} y) &= 0 \\ (h h^{-1} x_0 y) + (h x_1 h^{-1} y) &= 0 \\ x_0 y &= -(h x_1 h^{-1} y) \\ &\Rightarrow \\ g &= h v_i h^{-1} w_j \\ g w_j^{-1} &= h v_i h^{-1} \end{aligned}$$

This shows that $g w_j^{-1}$ is conjugate to something. Indeed, it must be that $h = g_{ij}$, so we're done as $e = h g_{ij}^{-1} \in C(v_i)$. Thus, $h \in C(v_i) g_{ij}$.

So, $C \subseteq \bigcup_{i=1}^{s} \bigcup_{j=1}^{m} C(v_i) g_{ij}$, and $C(v_i)$ must be finite, contradicting that $v_i \in \Delta(G)^C$. Thus, $0 = x_0 y = \pi(x) y$. □

3. Primitivity Theorems

We know by Connell's Theorem that $\Delta^+(G) = 1 \Leftrightarrow K[G]$ is a prime ring. We also know that a ring being primitive implies that it is prime. So, if we knew that $K[G]$ was primitive, is there something stronger we can prove? Indeed, there is.

Proposition 3.1 **Necessary Conditions**

Suppose that $K[G]$ is primitive with $|K| > |G|$. Then $\Delta(G) = 1$.

We need a few more results before we can prove this theorem. We will blackbox a few things relating to these results as they are technical in nature. There are 'two' steps.

1. $Z(K[G]) = K$ under our hypothesis.
2. $Z(K[G]) = \text{span}_K\left\{z_C := \sum_{c\in C} c \mid C \text{ is a finite conjugacy class}\right\}$

This shows that K is the span of finite conjugacy classes. But $K = \text{span}_K\{1_G\}$. So, every finite conjugacy class must be trivial, so $\{1_G\}$ is the only finite conjugacy class.

We can prove the second one easily.

Lemma 3.2 **The Center Is Sums of FC**

$Z(K[G]) = \text{span}_K\left\{z_C := \sum_{c\in C} c \mid C \text{ is a finite conjugacy class}\right\}$

Proof of The Center Is Sums of FC:

We do a double subset inclusion.

$\supseteq$: Let z_C be defined as above. We want to show it's central. We have:

$$\begin{aligned} z_C g &= \left(\sum_{c\in C} c\right) g \\ &= \left(\sum_{c\in C} gg^{-1}c\right) g \\ &= g\left(\sum_{c\in C} g^{-1}cg\right) \\ &= g\left(\sum_{c'\in C} c'\right) \\ &= gz_C \end{aligned}$$

$\subseteq$: Let $z = \sum \alpha_i z_i$ so that $z \in Z(K[G])$. Indeed, $g^{-1}zg = z \implies \sum \alpha_i z_i = \sum \alpha_i g^{-1} z_i g$.

Thus, $\alpha_i g^{-1} z_i g = \alpha_j z_j \implies g^{-1} z_i g = z_j$. But since z must be finitely supported, there can only be finitely many ways for z_i to map to the conjugacy class of z_j. Thus, each z_i belongs to a finite conjugacy class. Thus:

$$z \in \text{span}_K\{z_i\} \subseteq \text{span}_K\{z_C \mid C \text{ is a finite conjugacy class}\}$$

□

Proving the first step is quite difficult. In Passman's original paper, he deduces that we can replace K with $\overline{K}$ and the result will remain the same. However, it can be shown that

$Z(K[G])$ is algebraic, so regardless of which method one uses, the result remains true. We will follow Passman and assume K is algebraically closed.

Lemma 3.3 **The Center Is K**

Suppose that $K[G]$ is primitive with $|K| > |G|$. Additionally, assume that $K = \overline{K}$. Then, $Z(K[G]) = K$.

Proof of The Center Is K:

Notice that since K is commutative, we have that $K \subseteq Z(K[G])$ (where we abuse notation and let $K = K \cdot 1_G$). We need to show that $Z(K[G]) \subseteq K$.

First, we'll show that $Z(K[G])$ is a domain. Let M be the faithful simple module of $K[G]$. By Schur's Lemma, $D := \text{End}\left({}_{K[G]}M\right)$ is a division ring over $K[G]$. Notice that $Z(K[G]) \hookrightarrow D$ by mapping $z \mapsto f_z$, the scaling map by z. Also, z is central. Thus, $Z(K[G])$ is a domain.

Now, notice that by Jacobson's Density Theorem, $Z(K[G])$ is a homomorphic image of a subalgebra of $K[G]$. So, we have the following chain:

$$\dim_K Z(K[G]) \leq \dim_K K[G] = |G| < |K|$$

We'll now show that $Z(K[G]) \subseteq K$. Assume not. Let $d \in Z(K[G]) - K$ be fixed. Consider the set $\{(d-a)^{-1} \mid a \in K\}$. Notice that since a is a free variable, the set has cardinality $|K|$. So, we get linear dependence over K.

Indeed, choose a_i, b_i so that:

$$\sum_{i \in I} b_i (d - a_i)^{-1} = 0$$

We can now clear denominators. Since d commutes with each a_i, we get a nonzero polynomial over K so that d is a root. But then $d \in \overline{K} = K$. So, $Z(K[G]) \subseteq K$.

□

And with that, we found a necessary condition for $K[G]$ to be primitive. We'll now state three more sufficient conditions for primitivity which we won't prove, from Passman's paper.

Proposition 3.4 **Sufficient Conditions 1**

Let G be a locally finite countable group so that $\Delta(G) = 1$. Also assume that $JK[G] = 0$. Then, $K[G]$ is primitive.

Proposition 3.5 **Sufficient Conditions 2**

Let G be a polycyclic group so that $\Delta(G) = 1$. Let the trancendance degree of K be greater than the rank of G, the number of infinite cyclic quotients in G. Then, $K[G]$ is primitive.

Proposition 3.6 **Sufficient Conditions 3**

Let G be a torsion free solvable group so that $\Delta(G) = 1$. There exists some $F \supseteq K$ so that $F[G]$ is primitive.

4. Bibliography

Primitive Group Rings by D.S. Passman, Pacific Journal of Mathematics, Vol 47, No 2, 1973: https://projecteuclid.org/journals/pacific-journal-of-mathematics/volume-47/issue-2/Primitive-group-rings/pjm/1102945883.pdf