Sedenions and Algebras

By: Fahad Hossaini

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Note: I really want you to keep thinking 'what could come next?' throughout the presentation. The whole point is we keep looking forward for new directions, so I really want you to 'feel' that!

Double It…

Our journey starts conspicuously, with the most common object in mathematics, the real numbers.

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Some guy was real smart, and said "*hey, we can't factor some quadratics because* $\sqrt{-1}$ *DNE.*" So, we added i and taadaa, we get $\mathbb{C}$.

Then Hamilton...

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And as the famous story goes, he was crossing a bridge, and found the identity that pulled it all together: $ijk = -1, i^2 = -1, j^2 = -1, k^2 = -1$.

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And as the famous story goes, he was crossing a bridge, and found the identity that pulled it all together: $ijk = -1, i^2 = -1, j^2 = -1, k^2 = -1$.

And we could have stopped this story in 1898 or whatever. But Cayley...

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And why stop there? Give up and give in to the doubling. 16, 32, 64, and keep going. Never stop.

And these are our Cayley (or Cayley-Dickson) Algebras, denoted A_n where the dimension is 2^n.

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Then they learn that the quaternions are non-commutative ($ab \neq ba$).

Then they learn that the octonions are non-associative ($(ab)c \neq a(bc)$).

Then they learn that the sedenions have zero divisors ($xy = 0$ doesn't mean $x = 0$ or $y = 0$).

Why Associativity?

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Fundamentally, we love binary operations. Take two things, make one thing.

Associativity tells us the order in which we do this does not matter.

Imagine Multiplication Being Non-Associative

What if $(5 \cdot 3) \cdot 7 \neq 5 \cdot (3 \cdot 7)$. See how that's a headache?

And almost everything is associative. We *love* associativity and everything that matters is associative.

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And almost everything is associative. We *love* associativity and everything that matters is associative.

So, we really don't like the octonions. But, ok, we'll deal with that. But zero divisors?

Why Zero Divisors are Bad

Zero divisors really break the nice algebraic properties we can try to glue together with our scraps.

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Let's start by generalizing associativity.

We say an algebra A is alternative if, for all $x, y \in A$, $(xx)y = x(xy)$

The octonions are alternative. The sedenions are not!

These Zero Divisors Break Everything

If $xy = 0$, with $x \neq 0$ and $y \neq 0$, then we might get that $(x^2)y \neq 0 = x \cdot (xy)$.

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Luckily, we do know that: $(xy)x = x(yx)$. This is called flexibility, and every Cayley algebra is flexible.

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If these zero divisors are so problematic, why not just study them?

Arithmetic in Sedenions

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We define a basis: $\{1, e_1, e_2, \cdots, e_{15}\}$ where $e_i^2 = -1$, and write sedenions as a combination of these elements.

Basis elements are commutative in addition and anti-commutative in multiplication, and everything commutes with real numbers.

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So, $e_1 + e_5 = e_5 + e_1$ and $e_5 e_1 = -e_1 e_5$.

Multiplication

But how does multiplication actually work? Thanks for asking! Behold:

The Treacherous Table of The Talk

$\cdot$	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6	e_9	$-e_8$	e_{11}	e_{10}	$-e_{13}$	e_{12}	e_{15}	$-e_{14}$
e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$	e_{10}	e_{11}	$-e_8$	$-e_9$	$-e_{14}$	$-e_{15}$	e_{12}	e_{13}
e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$	e_{11}	$-e_{10}$	e_9	$-e_8$	$-e_{15}$	e_{14}	$-e_{13}$	e_{12}
e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3	e_{12}	e_{13}	e_{14}	e_{15}	$-e_8$	$-e_9$	$-e_{10}$	$-e_{11}$
e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2	e_{13}	$-e_{12}$	e_{15}	$-e_{14}$	e_9	$-e_8$	e_{11}	$-e_{10}$
e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$	e_{14}	$-e_{15}$	$-e_{12}$	e_{13}	e_{10}	$-e_{11}$	$-e_8$	e_9
e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1	e_{15}	e_{14}	$-e_{13}$	$-e_{12}$	e_{11}	e_{10}	$-e_9$	$-e_8$
e_8	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_{12}$	$-e_{13}$	$-e_{14}$	$-e_{15}$	-1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_9	e_8	$-e_{11}$	e_{10}	$-e_{13}$	e_{12}	e_{15}	$-e_{14}$	$-e_1$	-1	$-e_3$	e_2	$-e_5$	e_4	e_7	$-e_6$
e_{10}	e_{11}	e_8	$-e_9$	$-e_{14}$	$-e_{15}$	e_{12}	e_{13}	$-e_2$	e_3	-1	$-e_1$	$-e_6$	$-e_7$	e_4	e_5
e_{11}	$-e_{10}$	e_9	e_8	$-e_{15}$	e_{14}	$-e_{13}$	e_{12}	$-e_3$	$-e_2$	e_1	-1	$-e_7$	e_6	$-e_5$	e_4
e_{12}	e_{13}	e_{14}	e_{15}	e_8	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_4$	e_5	e_6	e_7	-1	$-e_1$	$-e_2$	$-e_3$
e_{13}	$-e_{12}$	e_{15}	$-e_{14}$	e_9	e_8	e_{11}	$-e_{10}$	$-e_5$	$-e_4$	e_7	$-e_6$	e_1	-1	e_3	$-e_2$
e_{14}	$-e_{15}$	$-e_{12}$	e_{13}	e_{10}	$-e_{11}$	e_8	e_9	$-e_6$	$-e_7$	$-e_4$	e_5	e_2	$-e_3$	-1	e_1
e_{15}	e_{14}	$-e_{13}$	$-e_{12}$	e_{11}	e_{10}	$-e_9$	e_8	$-e_7$	e_6	$-e_5$	$-e_4$	e_3	e_2	$-e_1$	-1

Working Inductively

We can take elements in A_{n-1} and build elements in A_n. For example, if $n = 4$, then we can take a pair of octonions and build a sedenion as follows: $(x, y) = x + e_8 y$ with $x, y \in A_3$.

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To formally define multiplcation, we define an operation called involution as well. We say $(x, y)^* = (a^*, -b)$. In $\mathbb{R} = A_0$, involution does nothing. In $\mathbb{C} = A_1$, involution is conjugation. Turns out, we just flip the sign for all the imaginary components.

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Then, if $a, b, c, d \in A_{n-1}$, then $(a, b) \cdot (c, d) = (ac - d^* b, da + bc^*)$.

We won't ever invoke this definition, but it explains why the multiplcation table can get so messy, especially as dimensions keep increasing.

Examples

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Alright, let's play a new game called: How many zero divisors.

How Many Zero Divisors?

So, $(e_3 + e_{10})(x + y) = e_3x + e_{10}x + e_3y + e_{10}y$. Ideally, $e_3x = -e_{10}y$. Let's see what pops up!

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And since multiplication by real numbers is linear, in fact, we get a linear subspace of zero divisors if we view the sedenions as an $\mathbb{R}$-vector space.

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Somehow, we have fallen back into a recognizable country, linear algebra.

The Structure of the Sedenions

Annihilators

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The proof of the theorem is quite long, but elementary. The real problem is, someone already discovered this. What's next?

One direction we might want to investigate is: What does multiplication *really* look like in the sedenions?

Well, if we had an automorphism, or, a map that goes from $A_4 \to A_4$ so that $f(xy) = f(x)f(y)$, then, we can (hopefully) get a better grasp on the multiplication of the sedenions.

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We know that multiplication in the quaternions are associative. So, if we take any two elements in the octonions, and consider the subspace spanned by those two elements, we should get something that looks similar to the quaternions.

Let's formalize this idea!

Generating Subspaces

Let $e_i, e_j \in A_3$ so that $e_i \neq e_j$. Then, $\text{span}\{1, e_i, e_j, e_i \cdot e_j\}$ is a subspace that's isomorphic to A_2.

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Can we try to do something similar, but with triples?

It turns out, in some sort of bizarre way, certain triples encode information about the sedenions.

Preserving Multiplication

Let e_i, e_j, e_k be denoted by (ijk). Then, the following triples encode multiplcation:

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In particular, define some $\varphi : A_3 \times A_3 \times A_3 \to \mathbb{R}$ by $\varphi(x, y, z) = \langle x \times y, z\rangle$ where this is the cross product and inner product. Indeed, this map secretly encodes multiplication.

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Given some map of A_3 to itself, say g, we want that $g^* \varphi = \varphi$.

Now, define $e^i(a)$ to be the coefficient of a in the ith basis element. Define:

$$e^{ijk}(x, y, z) = \det \begin{pmatrix} e^i(x) & e^i(y) & e^i(z) \\ e^j(x) & e^j(y) & e^j(z) \\ e^k(x) & e^k(y) & e^k(z) \end{pmatrix}$$

Here's the reveal, we can rewrite φ so that it remains unaffected by g as follows:

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But how does this help us?

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So, once again, the question we want to ask is, how does this help us? What's next?

There's actually not much left to do here. Trying to create injective or surjective homomorphisms from A_n to A_m doesn't help us 'learn more' about the sedenions.

We need a new idea.

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He tries to generalize associativity, links different triples together, shows what types of triples are reachable and does this all the way upto A_{10}.

Finally, he relates his structures with more definitions to classify zero divisors in as little elements as possible.

What Have We Learned?

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There are a few takeaways that we realized from trying to work with the sedenions.

- Zero divisors could be understood in certain contexts. But they usually really are hard to deal with.
- This idea of ‘doubling it’ is quite interesting. Is there somehow we can work with that in a general perspective?
- That element e_8 was quite interesting. Is there anything we can do to ‘imitate’ an element like that?
- Working with upto three elements seems to be the most we can work with before we lose too much generality.

Algebras

Why Algebras?

The idea is simple: If the sedenions are an algebra, why not take some bizarre properties of these sedenions and see if we can observe or use them to do something in different algebras.

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Any sort of classification is interesting, so why not classify algebras?

Formally Defining an Algebra

We define an algebra A as a vector space over some field $\mathbb{F}$ with some operation called multiplcation: $A \times A \to A$ by $(x, y) \mapsto xy$.

Notice that A only has addition and scaling, and no multiplcation if the dimension of A over $\mathbb{F}$ is greater than one.

This operation has only one restriction: Distributivity. So, if $a, b \in \mathbb{F}$ and $x, y, z \in A$, then:

$$(ax + by)z = a(xz) + b(yz) \mid x(ay + bz) = a(xy) + b(xz)$$

Formally Defining an Algebra

We define an algebra A as a vector space over some field $\mathbb{F}$ with some operation called multiplcation: $A \times A \to A$ by $(x, y) \mapsto xy$.

Notice that A only has addition and scaling, and no multiplcation if the dimension of A over $\mathbb{F}$ is greater than one.

This operation has only one restriction: Distributivity. So, if $a, b \in \mathbb{F}$ and $x, y, z \in A$, then:

$$(ax + by)z = a(xz) + b(yz) \mid x(ay + bz) = a(xy) + b(xz)$$

Notice that since we are working in a vector space, we have a basis and everything can be expressed linearly.

A Note on Field Choice

While an algebra is defined over any field, there are two common choices for this field. $\mathbb{R}$ and $\mathbb{C}$, as both are relatively nice.

In our case, we'll be working in $\mathbb{R}$ as this is nice to work with, and we need every advantage we can get (as you will come to see).

Linearity

In particular, let $\mathbb{F} = \mathbb{R}, a, b = 1, x, y, z \in A$ be basis vectors. Then:

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In particular, let $\mathbb{F} = \mathbb{R}$, $a, b = 1$, $x, y, z \in A$ be basis vectors. Then:

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Thus, we can extend multiplcation as long as we know how the basis vectors interact. Hence, why a multiplcation table is sufficient!

Once again, be careful not to conflate multiplcation in $\mathbb{F}$ with the operation we are defining.

Example

We can imagine $\mathbb{C}$ as a 2-dimensional $\mathbb{R}$-algebra with the following table:

$\cdot$	1	α
1	1	α
α	α	-1

Indeed, the sedenions are a 16-dimensional real algebra.

Multiplication Tables and Structure Constants

We can encode multiplication tables in a matrix as follows:

Write $e_i \cdot e_j = \sum_{k=0}^{n} a_{i,j} e_k$. Notice that $a_{i,j}$ are arbitrary coefficients. To associate $a_{i,j}$ to the basis element e_k, we'll classify it further by $a^k_{i,j}$. This is called a structure constant.

Now, for an n-dimension algebra, consider the matrix of dimenions $n \times n^2$:

$$\begin{pmatrix} a^1_{1,1} & a^1_{1,2} & \dots & a^1_{1,n} & a^1_{2,1} & \dots & a^1_{n,n} \\ a^2_{1,1} & a^2_{1,2} & \dots & a^2_{1,n} & a^2_{2,1} & \dots & a^2_{n,n} \\ \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ a^n_{1,1} & a^n_{1,2} & \dots & a^n_{1,n} & a^n_{2,1} & \dots & a^n_{n,n} \end{pmatrix}$$

Each column taken as a linear combination represents one 'cell' in the multiplcation table.

Where it All Breaks Down

The problem is this: When are two algebras isomorphic (They actually represent the same multiplcation)? Because any valid multiplcation table should yield an algebra.

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Attempting such a task for 3 dimensions is quite daunting, we go from 8 free variables to 27 free variables.

The most common tactic is to enforce constraints on our algebra, such as commutativity or division, and so on.

Known Results

Frobenius' Theorem (1877) states that any associative and division algebra is either $\mathbb{R}$, $\mathbb{C}$ or A_2.

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Any other classification aren't named theorems, or require a lot of prerequisite knowledge to understand.

Known 2 Dimensional Results

If we force A to be unitary (there exists some element $a \in A$ so that $ax = xa = x$ for all $x \in A$) and associative, we get three possible algebras:

$\cdot$	1	α
1	1	α
α	α	-1

$\cdot$	1	α
1	1	α
α	α	0

$\cdot$	1	α
1	1	α
α	α	1

These are your complex numbers, dual complex numbers and split complex numbers, in that particular order.

Full Classification

For general classifications of 2-dimensional real algebras, we have two results:

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Bekbaev (2017) classifies these by their structure constants and doing so allows us to impose stricter conditions if we want these algebras to be division or commutative.

In particular, Bekbaev finds 15 isomorphism classes with varying amounts of free variables. For example, he puts the following into one isomorphism class:

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 1 & 0 & -1 \end{pmatrix}$$

with $\beta_1 \in \mathbb{R}$ as a free choice.

Bringing it All Together

There are a few properties of the sedenions that don’t see use in any of the existing literature with regards to classifications.

Alter Scalars

An alter scaler is an element y so that for all $x \in A$, we have:

$$x^2(y) = x(xy)$$

In some ways, an alter-scalar is an element that results in a weak generalization of associativity.

In A_4, e_8 is an alter-scalar.

$\mathbb{Z}_2$ Grading

If we divide our multiplication table into disjoint halves, or disjoint equally sized subalgebras, say A_1, A_2 so that $A_1 \cap A_2 = \emptyset$, and $\dim A_0 = \dim A_1$, then we say the algebra is $\mathbb{Z}_2$ graded if:

$$A_0 \times A_0 \subset A_0, A_1 \times A_1 \subset A_0, A_1 \times A_0 \subset A_1, A_0 \times A_1 \subset A_1$$

Or,

$\cdot$	A_0	A_1
A_0	A_0	A_1
A_1	A_1	A_0

By Bekbaev, given a matrix of structure constants, we can produce a formula to determine if the algebra is a division algebra or not. It's quite messy, so I won't explain it.

Given, $\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix}$, define:

$$\Delta_a = \det\left(\begin{pmatrix} \alpha_1 & \alpha_3 \\ \beta_1 & \beta_3 \end{pmatrix}\right), \Delta_b = \det\left(\begin{pmatrix} \alpha_1 & \alpha_4 \\ \beta_1 & \beta_4 \end{pmatrix}\right) + \det\left(\begin{pmatrix} \alpha_2 & \alpha_3 \\ \beta_2 & \beta_3 \end{pmatrix}\right), \Delta_c = \det\left(\begin{pmatrix} \alpha_2 & \alpha_4 \\ \beta_2 & \beta_4 \end{pmatrix}\right)$$

Then, the algebra is division if and only if $D = \Delta_b^2 - 4\Delta_a\Delta_c < 0$

Where We Are in The Current Day

We're trying to combine these ideas alongside some other ones to reach some classification result.

In particular, $\mathbb{Z}_2$ grading forces that our classification results are 4 dimensional.

Another idea we're experimenting with is trying to related different dimension algebras to one another.

Thank You!