The Artin-Wedderburn Theorem

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01

Rings And Fields

Remember fields?

1.1 What is a ring?

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Axioms	**Fields**	**Rings**	**Rings (cool)**
Associativity (+)	✅	✅	✅
Associativity $(\cdot)$	✅	✅	✅
Identity (+)	✅	✅	✅
Identity $(\cdot)$	✅	✅	✅
Inverses (+)	✅	✅	✅
Inverses $(\cdot)$	✅	❌	❌
Distributivity	✅	✅	✅
Commutativity (+)	✅	✅	✅
Commutativity $(\cdot)$	✅	✅	❌

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🏁 **Goal**

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- What is a "nice" ring?
- How do we even begin to examine rings?
- What properties lead to a classification?

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5. $C^0(\mathbb{R})$ - continuous functions of the form $f : \mathbb{R} \to \mathbb{R}$

02

Modules

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What if $\mathbb{Q}$ is over $\mathbb{Z}$? This would be infinite dimensional!

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There are *infinitely* many submodules - we can even make infinite 'chains' in this case. Vector spaces don't do that!

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2.4 Comparison Game

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Dimension	Solid, reliable	What, the dream dimension?

Worse - even if distinct bases exist, they can be of different cardinality!

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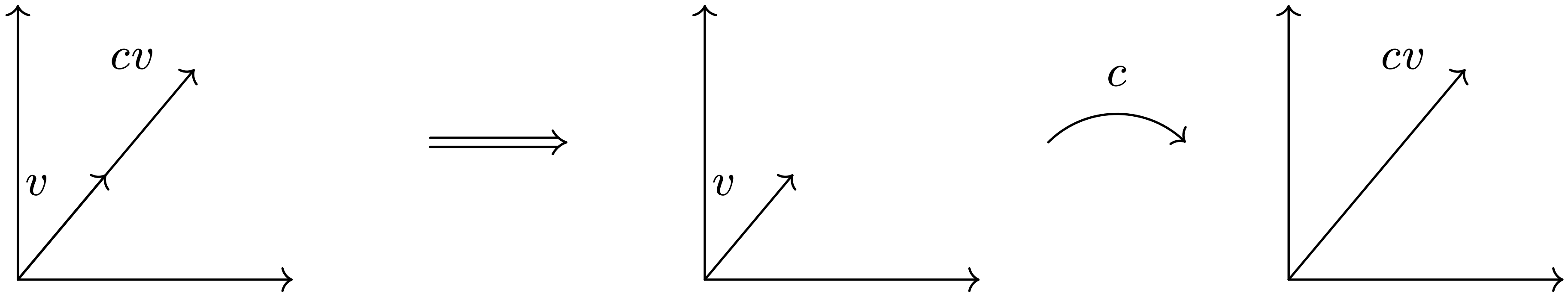
Alt perspective: if V is a set with addition and $c \in \mathbb{R}$, interpret cv not as v "scaled" by c, but instead a map $\phi_c : V \to V$ given by $v \mapsto cv$.



Special thanks to M. Motiwala for the code for this diagram!

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Analyze scaling on module $\implies$ learn stuff about R!

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We'll revisit this analogy a lot.

(See board for visual diagram)

03

(Semi-)Simplicity

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  - Proof: Take a basis $\{e_i\}$. Each $M_i = \operatorname{span}(e_i)$ is simple, and $V = \bigoplus_i M_i$.

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We'll take these as our definition of "nice" rings.

04

Annihilators

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In this case, the annihilator is (6).

Think of a flashlight reducing a pencil to a point.

We define the **annihilator** of a module M over a ring R as the elements of R which "annihilate" M in the previously mentioned bad sense.

$$\mathrm{Ann}_R(M) := \{r \in R : rm = 0 \text{ for all } m \in M\}$$

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- Suppose $Av = 0$ for all v. Then $Ae_1 = Ae_2 = 0$.
- But Ae_k is the k-th column of A, so A is the zero matrix!

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$$\mathbb{Z}_3 \text{ is simple} \quad , \quad 3\mathbb{Z}_3 = 0 \quad , \quad (3) = \{\text{multiples of } 3\} \quad \Longrightarrow \quad (3) \text{ is primitive in } \mathbb{Z}$$

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We say a ring R is **primitive** if $\{0\}$ is primitive in R. Equivalently, there exists a *faithful simple module* over R.

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- Recall $\mathbb{R}^2$ is faithful over $M_2(\mathbb{R})$.
- It is also simple, since any non-zero vector can get mapped to any vector.
- Thus, $M_2(\mathbb{R})$ has a faithful simple module - i.e. it is primitive!

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We say a ring R is **semiprimitive** if $J(R) = \{0\}$. In other words, it has no blind spot.

Think of a cube with a ball in it - this has a blind spot.

05

Tying It All Together

That was a lot of definitions! Let's look at some examples.

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How do these definitions relate?

We'll show that semisimple implies semiprimitive.

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Our flashlights are: $\mathbb{Z}_2 \times \{0\}$ as a $\mathbb{Z}_6$-module and $\{0\} \times \mathbb{Z}_3$ as a $\mathbb{Z}_6$ module.

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In particular, we can 'choose' a smart chain. For example,

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In other words, we can't escape to infinity, since the direct product must be finite.

So, we have the following theorem:

$$R \text{ is semisimple} \Leftrightarrow J(R) = 0 \text{ and } R \text{ is artinian}$$

Or, we can write a ring as a direct product if nothing is ever hidden by choosing all possible flashlights, and we only need finitely many flashlights.

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But what are these rings? Is there any classification theorem that classifies every ring that looks like this?

06

The Artin-Wedderburn Theorem

Every semisimple ring is a direct product of matrix rings over division rings.

We'll prove this in two steps.

1. We'll show that when the ring is primitive and artinian, this is not a product, but just a single matrix ring over a division ring.
2. We'll now add in semiprimitivity, which now means our ring is semisimple. Then, we can show that this addition only makes it a product now.

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Proof: R primitive $\Rightarrow$ faithful simple module M. We'll prove $\Delta := \operatorname{End}({}_R M)$ is a division ring. This is Schur's Lemma.

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Thus, we get the following:

$$\text{Semiprimitive} \Rightarrow (0) = \bigcap \mathcal{m}_i = J(R)$$

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We'll invoke an ancient theorem, called Sunzi's Theorem. This is actually just the Chinese Remainder Theorem.

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$$\begin{aligned} R &\simeq R/(0) \\ &\simeq R/J(R) \\ &\simeq R/\bigcap m_i \\ &\simeq R/m_1 \times \cdots \times R/m_k \end{aligned}$$

The upshot: Each $R/\mathfrak{m}_i$ is a primitive and artinian ring.

But by AWT, each $R/\mathfrak{m}_i$ is a matrix ring over some division ring!

Thus:

$$R \simeq R/\mathfrak{m}_1 \times \cdots \times R/\mathfrak{m}_k \simeq M_{n_1}(\Delta_1) \times \cdots \times M_{n_k}(\Delta_k)$$

∎

07

End Of Slidedeck!

Thank You!