p-adic Analysis

p-adic and Their Problematic Antics Part 2

Mustafa Motiwala And Fahad Hossaini

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The motivating question

How can we measure the size of rational numbers?

Definition. An absolute value on a field $\mathbb{F}$ is a function $|\cdot| : \mathbb{F} \to \mathbb{R}$ satisfying, for all $x, y \in \mathbb{F}$:

1. $|x| \geq 0$ with $|x| = 0$ iff $x = 0$.
2. $|xy| = |x||y|$.
3. $|x + y| \leq |x| + |y|$.

The "normal" absolute value on $\mathbb{Q}$ measures size by physical distance to 0. How about something more algebraic?

The p-adic valuation

By the FTA, every non-zero integer $n \in \mathbb{Z}$ factors uniquely as a product of primes. If we allow exponents to be negative, this expression lifts to $\mathbb{Q}$. Thus, for any $0 \neq x \in \mathbb{Q}$, we can write x uniquely as

$$x = \pm \prod_{p \text{ prime}} p^{v_p(x)}$$

For example, $\frac{2468}{4567} = 2^2 \cdot 617 \cdot 4567^{-1}$ so $v_2(\frac{2468}{4567}) = 2$ and $v_{4567}(\frac{2468}{4567}) = -1$.

Conventionally, we take $v_p(0) = \infty$. The valuation measures divisibility by the prime p.

The p-adic absolute value

The valuation v_p measures divisibility by p. To turn this into an absolute value, we exponentiate, and since 0 is infinitely divisible by p, the more factors of p a number has, the closer it should get to 0. So, we arrive at the following definition.

Definition. Let p be prime. We define the p-adic absolute value on $\mathbb{Q}$ by

$$|x|_p = p^{-v_p(x)}$$

Remark. Up to induced topology, these are the only other absolute values on $\mathbb{Q}$!

$\mathbb{Q}_p$

Under the standard absolute value, we have sequences of rational numbers like $3, 3.1, \dots, 3.14159265, \dots$ that get closer to each other but don't converge to a rational number. We say $\mathbb{Q}$ is "incomplete" under this absolute value.

Something similar happens with the p-adic absolute value. For example, under the 7-adic absolute value, there is a sequence $2, 30, 324, 1696, \dots$ that converges to $\sqrt{18}$.

Just like we can construct $\mathbb{R}$ from $\mathbb{Q}$ by "filling in the gaps" and making it complete with respect to the standard absolute value, we can enlarge $\mathbb{Q}$ to be complete with respect to the p-adic absolute value. The resulting set is called $\mathbb{Q}_p$: the smallest complete field extending the p-adic absolute value on $\mathbb{Q}$.

Wait, so what actually is $\mathbb{Q}_p$?

Theorem. Let $x \in \mathbb{Q}_p$. Let $n_0 = v_p(x)$. There exists a unique sequence $d_{n_0}, d_{n_0+1}, \ldots$ with each $d_k \in \{0, \cdots, p-1\}$ such that $x = \sum_{k \geq n_0} d_k p^k$.

Sound familiar? What this really means is every element in $\mathbb{Q}_p$ has a unique base-p expansion that extends infinitely to the left! The following are all elements of $\mathbb{Q}_5$:

- $\ldots 0234_5$
- $\ldots 44444_5$
- $\ldots 1.111_5$
- $\ldots 0234.314_5$
- $\ldots 112340_5$
- $\ldots 0_5$

Arithmetic

We can exploit the base-p expansion of p-adics to compute with them. The nice part is everything works exactly the same as you're used to! For example,

$$\begin{array}{r} \dots 0234_5 \\ + \dots 0111_5 \\ \hline \dots 0400_5 \end{array}$$

Of course, each number is infinite, so this is in general an infinite process, but each digit can be computed in finite time.

Some code

For the purposes of computing with some $x \in \mathbb{Q}_p$, we need only know n_0, the smallest power of p appearing in the expansion of x, and the digits of x. In (Haskell) code, we have

```
data PAdic a = PAdic {
          p :: Integer
        , n0 :: Integer
        , digits :: [a]
    }
```

Addition

Let $x = \ldots x_2 x_1 x_0$, $y = \ldots y_2 y_1 y_0$ and write $r = x + y = \ldots r_2 r_1 r_0$. We want to solve for each r_k. Let c_k denote the carry after computing r_k.

By the gradeschool algorithm, $r_0 = (x_0 + y_0) \mod p$ and $c_0 = \left\lfloor \frac{x_0+y_0}{p} \right\rfloor$ and

$$r_{n+1} = (x_{n+1} + y_{n+1} + c_n) \mod p$$
$$c_{n+1} = \left\lfloor \frac{x_{n+1} + y_{n+1} + c_n}{p} \right\rfloor$$

```
(PAdic p n0 ds0) + (PAdic p n1 ds1)
    = PAdic p n $ go 0 (grow n ds0) (grow n ds1) where
      n = min n0 n1
      go c (a:as) (b:bs)
          = let r = a + b + c in
            (r `mod` p) : go (r `div` p) as bs
```

Negation

Suppose we have a p-adic number $x = \dots x_2 x_1 x_{0p}$. What is its negation? Let's write $y = -x = \dots y_2 y_1 y_{0p}$ and solve for each y_k.

We need $x_0 + y_0 = 10_p = p$, so $y_0 = -x_0 \mod p$.

Then, we have a carry digit, so we need $x_1 + y_1 + 1 = 10_p = p$ and hence $y_1 = -(x_1 + 1) \mod p$.

The pattern continues from here, so we have

$$y_k = \begin{cases} p - x_0 & k = 0 \\ p - 1 - x_k & k > 0 \end{cases}$$

In code,

```
negate x
    = x { digits = (p - d):map ((p-1)-) ds } where
      PAdic p _ (d:ds) = x
```

Multiplication

Let $x = \dots x_2 x_1 x_0$, $y = \dots y_2 y_1 y_0$ and write $r = x \times y = \dots r_2 r_1 r_0$. Officially, the gradeschool algorithm says

$$r = y_0 x + y_1 p x + \cdots + y_k p^k x + \cdots$$

Now, the first digit of the sum, r_0, will simply be the first digit of $y_0 x$, since everything else is multiplied by p. Likewise, for the second digit, we need only know $y_0 x, y_1 x$. As it happens, with c_k as before, we have

$$r_k = \left(c_{k-1} + \sum_{i+j=k} x_i y_j \right) \mod p$$

$$c_k = \left\lfloor \frac{c_{k-1} + \sum_{i+j=k} x_i y_j}{p} \right\rfloor$$

Multiplication (code)

```
(PAdic p n0 ds0) * (PAdic p n1 ds1)
    = PAdic p n $ go 0 [] ds1 where
        n = n0 + n1

        go :: Integer -> [[Integer]] -> [Integer] -> [Integer]
        go c s (b:bs)
            = let s' = map (b*) ds0 : s
                  r = c + sum (map head s')
                  d = r `mod` p
                  c' = r `quot` p
              in
                d : go c' (map tail s') bs
```

Division

Let $x, y \in \mathbb{Q}_p$. To compute $r = x/y$, we "compute" $x = ry$ (which we know how to do) and solve for each digit of r on the way. Since we already know how to multiply, we might as well focus on the case $x = 1$ (so that really we're computing the reciprocal). In code,

```
recip x@(PAdic _ _ (a0:_))
    | a0 == 0 = recip (simplifyN0 x)
recip (PAdic p n0 (a0:as)) = PAdic p (-n0) $ go 0 [] target where
    -- a0 = as !! 0
    a0inv = modularInverse a0 p
    target = 1:repeat 0

    go :: Integer -> [[Integer]] -> [Integer] -> [Integer]
    go c s (t:ts) =
        let d = (a0inv * (t - c - sum (map head s))) `mod` p
            c' = (c + sum (map head s) + d * a0) `div` p
            s' = map (d*) as : map tail s
        in
            d:go c' s' ts
```

Something's up with the p-adic absolute value...

We said that any absolute value needs to satisfy $|x+y| \leq |x| + |y|$. Does the p-adic absolute value meet this condition? Let's check.
Say $p = 2, x = 10, y = 32$. Then, $|10|_2 = 1/2, |32|_2 = 1/32$, and $|10+32|_2 = |42|_2 = 1/2$. Hm... let's try another example.

Take $p = 5, x = 20, y = 5$. Then $|20|_5 = 1/5, |5|_5 = 1/5$, and $|20+5|_5 = 1/25$.

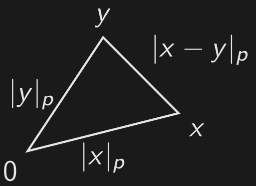
It seems that not only do we have $|x+y|_p \leq |x|_p + |y|_p$, we have $|x+y|_p \leq \max\{|x|_p, |y|_p\}$. In fact, this is always true. We say the p-adic absolute value is *non-archimedean* because for all $x, y \in \mathbb{Q}_p$, we have

$$|x+y|_p \leq \max\{|x|_p, |y|_p\}$$

Non-Archimedeanness is weird

Consider a triangle in $\mathbb{Q}_p$. WLOG, we put one vertex on the origin, and call the other two vertices $x, y \in \mathbb{Q}_p$.

Suppose for the moment that $|x|_p \neq |y|_p$; we might as well assume $|x|_p > |y|_p$ (otherwise, rename x, y). What is $|x-y|_p$? On the one hand, we know $|x-y|_p \leq \max\{|x|_p, |y|_p\} = |x|_p$, by non-archimedeanness.



On the other hand, $|x|_p = |(x-y)+y|_p \leq \max\{|x-y|_p, |y|_p\}$. Since $|x|_p > |y|_p$, we must have $|x|_p \leq |x-y|_p$.

Both $|x-y|_p \leq |x|_p$ and $|x|_p \leq |x-y|_p$ give $|x|_p = |x-y|_p$.

Non-Archimedeanness is weird (cont.)

That is, take any two sides of a triangle in $\mathbb{Q}_p$: if they're not equal in length, then the longer one has the same length as the remaining side. In other words, **all triangles in $\mathbb{Q}_p$ are isosceles!**. We'll refer to this surprising fact as ATAI.

Let's say this again in another form that'll often be useful.

Theorem. Let $x_1, x_2, \dots \in \mathbb{Q}_p$. Suppose $N \in \mathbb{N}$ is such that $|x_N|_p > |x_k|_p$ for all $k \neq N$. Then, assuming everything converges, $\left|\sum_{k=0}^{\infty} x_k\right|_p = |x_N|_p$.

The takeaway: we can compute the absolute value of a sum without needing to compute the sum! I know without needing to do any calculation that $|1 + 1/11 + 1/121|_{11} = 121$.

Balls in $\mathbb{Q}_p$

Given $a \in \mathbb{Q}_p$ and $r \geq 0$, we define the "open" and "closed" balls as

$$B(a, r) = \{x \in \mathbb{Q}_p : |x - a|_p < r\}$$
$$\overline{B}(a, r) = \{x \in \mathbb{Q}_p : |x - a|_p \leq r\}$$

These balls generate the topology on $\mathbb{Q}_p$ so studying them makes sense. Turns out, they're pretty weird!

Balls don't have unique centres

Let $B(a,r)$ be a ball in $\mathbb{Q}_p$. Let $b \in B(a,r)$ be any other point in the ball. It turns out b is also a centre of the ball; that is, $B(a,r) = B(b,r)$.

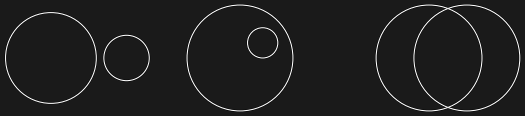
Indeed, suppose $b \in B(a,r)$ so that $|a-b|_p < r$. Then for any $x \in B(a,r)$, we have

$$|x-b|_p = |(x-a)+(a-b)|_p \leq \max\{|x-a|_p, |a-b|_p\} < r$$

so $x \in B(b,r)$. The reverse inclusion is identical, so $B(a,r) = B(b,r)$. Swapping $<$ for $\leq$ gives the same result for $\overline{B}(a,r)$.

Balls never have non-trivial overlap

In $\mathbb{Q}_p$, we can have



These are possible. . . but not this!

This follows from the previous slide. Suppose $B(a, r), B(b, s)$ have the point c in common. WLOG, $r \leq s$, so

$$B(a, r) = B(c, r) \subseteq B(c, s) = B(b, s)$$

as needed. Again, changing B for $\overline{B}$ gives the result for closed balls.

Open balls are closed

Let $B(a, r)$ be an open ball in $\mathbb{Q}_p$. Let

$$C = \mathbb{Q}_p \setminus B(a, r) = \{x \in \mathbb{Q}_p : |x - a|_p \geq r\}$$

We'll show C is open too. Take any $y \in C$ and any $s < r$. If $z \in B(y, s)$, then $|z - y|_p < s < |y - a|_p$ so by ATAI,

$$|z - a|_p = |(z - y) + (y - a)|_p = |y - a|_p \geq r$$

and hence $z \in C$. Thus, every $y \in C$ is the centre of some open ball contained in C, so C is open and hence $B(a, r)$ is closed.
If $r \neq 0$, a similar argument shows that closed balls are open.
So really, **all balls are clopen!**

$\mathbb{Q}_p$ is totally disconnected

Recall that a set is disconnected if it cannot be written as a union of disjoint, non-empty open sets. In $\mathbb{Q}_p$, the only connected sets are the singletons.

Suppose $S \subseteq \mathbb{Q}_p$ contains two distinct points $x \neq y$. Taking $r = |x-y|_p/2$, we see that $y \notin B(x,r)$ so $S = (B(a,r) \cap S) \cup (B(a,r)^c \cap S)$. Since balls are clopen, both sets are open (in S), so S is disconnected.

The Cauchy criterion

A *Cauchy sequence* is a sequence (a_n) such that $\lim_{n,m\to\infty} |a_n - a_m| = 0$ uniformly in n, m. In a complete space, a sequence converges if and only if it is Cauchy. In $\mathbb{R}$, this is not the same as consecutive terms getting close; if $a_n = 1 + 1/2 + \cdots + 1/n$, then $a_{n+1} - a_n = 1/(n+1) \to 0$ but the sequence (a_n) does not converge and hence is not Cauchy.

In $\mathbb{Q}_p$, though, the situation is different. We have

$$\begin{aligned}
\lim_{n,m\to\infty} |a_n - a_m| &= \lim_{n,m\to\infty} |a_n - a_{n+1} + a_{n+1} - a_{n+2} + \cdots - a_m| \\
&\leq \lim_{n,m\to\infty} \max\{|a_n - a_{n+1}|, \ldots, |a_{m-1} - a_m|\} \\
&= \lim_{n\to\infty} |a_n - a_{n+1}|
\end{aligned}$$

so a sequence in $\mathbb{Q}_p$ is Cauchy if and only if consecutive terms get close.

The MAT136 dream

Let's come back to $\mathbb{R}$ for just a second. It's obvious that if a series $\sum_k a_k$ converges, then we must have $a_k \to 0$. Alas, the converse is not true, because $1/k \to 0$ but the harmonic series $\sum_k 1/k$ famously diverges.

The situation is much nicer in $\mathbb{Q}_p$.

Theorem. Let $(a_k)_{k=1}^{\infty}$ be a sequence in $\mathbb{Q}_p$. Then, the series $\sum_{k=1}^{\infty} a_k$ converges *if and only if* $\lim_{k\to\infty} a_k = 0$.

Proof. The series converges iff the sequence $s_n = \sum_{k=1}^{n} a_k$ is Cauchy iff $\lim_{n\to\infty} |s_{n+1} - s_n| = 0$. But, $s_{n+1} - s_n = a_{n+1}$, so we have the result. $\square$

Why Series?

(Hopefully) you were wondering why we talked about series. There's actually a few big problems:

- Since $\mathbb{Q}_p$ is totally disconnected, we don't have IVT.
- We can have a non-constant function with derivative zero everywhere. (So no MVT)

These three things make formal derivatives 'useless.' What about integration?

An Aside On Integration

Integration doesn't behave nicely because we don't have a nice measure.
Consider the unit ball in $\mathbb{Q}_p$.

- Moving the ball to different centers shouldn't change its size.
- The size of the unit ball is a fixed amount.
- We determine size via the p-adic absolute value.

The point: This isn't possible. So integration is very tricky, and a very interesting subject.

Power Series

Our best bet is to work with functions defined by power series. Recall, a power series is a series of the following form:

$$f(x) = \sum_{i=0}^{\infty} a_n x^n$$

Where $a_n \in \mathbb{Z}_p$ and $x \in \mathbb{Q}_p$.

This function is defined for x when the power series converges. This is called our radius of convergence, usually denoted R.

e Doesn't Exist

One of the common functions we deal with is e^x. But $e \notin \mathbb{Q}_p$. So, we have no choice but to rely on the power series.
Recall,

$$exp(x) = \sum_{i=0}^{\infty} = 1 + x + \frac{x}{2!} + \frac{x}{3!} + \cdots$$

In $\mathbb{R}$, this has an infinite radius of convergence. What about in $\mathbb{Q}_p$?

Hold up... how do we calculate radius of convergence?

We'll employ the root test! Recall, R, the radius of convergence, is given by:

$$R = \frac{1}{limsup_{n\to\infty} \sqrt[n]{|a_n|}}$$

In our case, $a_n = \frac{1}{n!}$. Thus, $|a_n| = p^{v_p(n!)}$ So, what's $limsup_{n\to\infty} \sqrt[n]{|a_n|}$?

The exponent will be: $limsup_{n\to\infty} \frac{v_p(n!)}{n}$

Let's take a look at $p = 2$.

The Treacherous Table Of The Talk!

n	$v_p(n)$	$v_p(n!)$	$v_p(n!)/n$
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			

The Treacherous Table Of The Talk!

n	$v_p(n)$	$v_p(n!)$	$v_p(n!)/n$
1	0	0	0/1
2			
3			
4			
5			
6			
7			
8			
9			
10			

The Treacherous Table Of The Talk!

n	$v_p(n)$	$v_p(n!)$	$v_p(n!)/n$
1	0	0	0/1
2	1	1	1/2
3	0	1	1/3
4			
5			
6			
7			
8			
9			
10			

The Treacherous Table Of The Talk!

n	$v_p(n)$	$v_p(n!)$	$v_p(n!)/n$
1	0	0	0/1
2	1	1	1/2
3	0	1	1/3
4	2	3	3/4
5	0	3	3/5
6	1	4	4/6
7	0	4	4/7
8	3	7	7/8
9	0	7	7/9
10	1	8	8/10

The Treacherous Table Of The Talk!

n	$v_p(n)$	$v_p(n!)$	$v_p(n!)/n$
1	0	0	$0/1$
2	1	1	$1/2$
3	0	1	$1/3$
4	2	3	$3/4$
5	0	3	$3/5$
6	1	4	$4/6$
7	0	4	$4/7$
8	3	7	$7/8$
9	0	7	$7/9$
10	1	8	$8/10$
16	4	15	$15/16$
32	5	31	$31/32$

Convergence at $p = 2$

The point: While the limit doesn't exist, $limsup_{n\to\infty}\frac{v_p(n!)}{n} = 1$.

So, the radius of convergence of $exp(x)$ when $p = 2$ is $|x| \leq p^{-1}$.

Remember, the root test doesn't tell us anything about the endpoints! But an off-screen calculation shows us that at the endpoints, we have convergence!

Log And Games

Now, as a short intermission, we'll play a game of True and False!

But before we begin, let's recall the power series for *log*!

$$log(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n} = (x-1) - \frac{(x-1)^2}{2} + \cdots$$

Let's Play A Game

We'll assume that our power series converge. So, no 'tricks.'

$$exp(a+b) = exp(a)exp(b) \mid ??$$

Should I Switch The Slide Titles Everytime?

$$exp(a+b) = exp(a)exp(b) \;\Big|\; \text{True}$$

Let's Play A Game

$$\begin{array}{c|c} exp(a+b) = exp(a)exp(b) & \text{True} \\ log(ab) = log(a) + log(b) & ?? \end{array}$$

Let's Play A Game

$$\begin{array}{c|l} exp(a+b) = exp(a)exp(b) & \text{True} \\ log(ab) = log(a) + log(b) & \text{True} \end{array}$$

Let's Play A Game

$exp(a+b) = exp(a)exp(b)$	True
$log(ab) = log(a) + log(b)$	True
$exp(log(a)) = a$	??

Assume that power series composition behaves 'nicely.' So no tricks.

Let's Play A Game

$$\begin{array}{c|l} exp(a+b) = exp(a)exp(b) & \text{True} \\ log(ab) = log(a) + log(b) & \text{True} \\ exp(log(a)) = a & \text{True} \end{array}$$

Let's Play A Game

$$\begin{array}{c|c}
exp(a+b) = exp(a)exp(b) & \text{True} \\
log(ab) = log(a) + log(b) & \text{True} \\
exp(log(a)) = a & \text{True} \\
|exp(x) - exp(y)| < |x - y| & ??
\end{array}$$

Let's Play A Game

$exp(a+b) = exp(a)exp(b)$	True
$log(ab) = log(a) + log(b)$	True
$exp(log(a)) = a$	True
$\|exp(x) - exp(y)\| < \|x - y\|$	False

The following is true though: $|exp(x) - exp(y)| = |x - y|$

Let's Play A Game

$exp(a+b) = exp(a)exp(b)$	True
$log(ab) = log(a) + log(b)$	True
$exp(log(a)) = a$	True
$\lvert exp(x) - exp(y)\rvert < \lvert x - y\rvert$	False
$log(1) = 0$ is the only zero for all prime p	??

Let's Play A Game

$exp(a+b) = exp(a)exp(b)$	True
$log(ab) = log(a) + log(b)$	True
$exp(log(a)) = a$	True
$\lvert exp(x) - exp(y)\rvert < \lvert x - y\rvert$	False
$log(1) = 0$ is the only zero for all prime p	False

When $p = 2$, $log(-1) = 0$ since -1 is in our radius of convergence.

Additionally, these are the only zeroes of log!

Let's Play A Game

$exp(a+b) = exp(a)exp(b)$	True
$log(ab) = log(a) + log(b)$	True
$exp(log(a)) = a$	True
$\lvert exp(x) - exp(y)\rvert < \lvert x - y\rvert$	False
$log(1) = 0$ is the only zero for all prime p	False
Periodic functions (power series) are constant	??

Here, the periodic function is a power series with a finite radius of convergence.

Let's Play A Game

$exp(a+b) = exp(a)exp(b)$	True
$log(ab) = log(a) + log(b)$	True
$exp(log(a)) = a$	True
$\lvert exp(x) - exp(y) \rvert < \lvert x - y \rvert$	False
$log(1) = 0$ is the only zero for all prime p	False
Periodic functions (power series) are constant	True

Let's Play A Game

$exp(a+b) = exp(a)exp(b)$	True
$log(ab) = log(a) + log(b)$	True
$exp(log(a)) = a$	True
$\lvert exp(x) - exp(y)\rvert < \lvert x - y\rvert$	False
$log(1) = 0$ is the only zero for all prime p	False
Periodic functions (power series) are constant	True
$exp(x) = cos(x) + isin(x)$	??

Where *cos* and *sin* are defined by power series

Let's Play A Game

$exp(a+b) = exp(a)exp(b)$	True
$log(ab) = log(a) + log(b)$	True
$exp(log(a)) = a$	True
$\lvert exp(x) - exp(y)\rvert < \lvert x - y\rvert$	False
$log(1) = 0$ is the only zero for all prime p	False
Periodic functions (power series) are constant	True
$exp(x) = cos(x) + isin(x)$	True

Binomials

We can define the binomial series as follows:

$$B(\frac{a}{b}, x) = (1+x)^{\frac{a}{b}} = \sum_{n=0}^{\infty} \binom{a/b}{n} x^n$$

The simplest thing to look at, is when $(1+x^{\frac{1}{2}}$ is a rational number. Intuitively, we want to think of this as the square root, and ensure the binomial series align with our intuition.

So, let $(1+x)^{\frac{1}{2}} = \frac{a}{b}$. Rearranging, we get:

$$x = \frac{a^2}{b^2} - 1$$

When does this converge for different $\mathbb{Q}_p$?

An Example

Let's choose $a = 4, b = 3$. So, $\frac{a^2}{b^2} - 1 = \frac{16}{9} - 1 = \frac{7}{9} = x$.

In $\mathbb{R}$, we have that this $(1+x)^{\frac{1}{2}} = \frac{4}{3}$ by construction.

What about $\mathbb{Q}_7$? Well, $|(x+1)^{\frac{1}{2}} - 1|_7 < 1$.

But $|\frac{4}{3} - 1|_7 = |\frac{1}{3}|_7 = 1$

What Just Happened?

We showed that the binomial series in $\mathbb{Q}_7$ can't converge to $\frac{4}{3}$ or else we'd get a contradiction!

In fact, we have that $(x+1)^{\frac{1}{2}} = -\frac{4}{3}$.

So, even though we designed our series to converge to $\frac{4}{3}$, we can find some $\mathbb{Q}_p$ so that it converges to the negative!

Examination

Is this a one-off example?

To figure this out, we need to find out when and where $\mathbb{Q}_p$ converges to given some fraction $\frac{a}{b}$.

We have a really weird list of criteria to check for convergence.

Criteria

We have convergence in $\mathbb{Q}_p$ when:

- $p|(a+b)(a-b)$ for odd p.
- If a, b are both odd, then $p = 2$ converges.
- If $0 < \frac{a}{b} < \sqrt{2}$, then we have convergence in $\mathbb{R}$.

A Weird List

We can show that we must converge in some $\mathbb{Q}_p$ or $\mathbb{R}$.

But, furthermore, for 4 specific numbers, we only converge in some $\mathbb{Q}_p$ for a singular p, and have no convergence in $\mathbb{R}$.

A Weird List

We can show that we must converge in some $\mathbb{Q}_p$ or $\mathbb{R}$.

But, furthermore, for 4 specific numbers, we only converge in some $\mathbb{Q}_p$ for a singular p, and have no convergence in $\mathbb{R}$.

$$x = 8, 3, \frac{16}{9}, \frac{5}{4}$$

$x = 8$

When $x = 8$, we have that $a = 3$ and $b = 1$.

Clearly, no odd primes divide 2 or 4.

And $\frac{a}{b} > \sqrt{2}$ so no convergence in $\mathbb{R}$

So, this only converges in $\mathbb{Q}_2$.

The Setup

Let's create our table of a and b given x:

$x = 8$	$a = 3$	$b = 1$	$p = 2$
$x = 3$	$a = 2$	$b = 1$	$p = 2$
$x = \frac{16}{9}$	$a = 5$	$b = 3$	$p = 3$
$x = \frac{5}{4}$	$a = 3$	$b = 2$	$p = 5$

The Setup 2.0

We want to focus on a, b and we'll change the values a little.

$$
\begin{array}{r|r|r|r|c}
x = 8 & a = 3 & b = 1 & p = 2 & \frac{3}{1} \\
x = 3 & a = 2 & b = 1 & p = 2 & \frac{4}{2} \\
x = \frac{16}{9} & a = 5 & b = 3 & p = 3 & \frac{5}{3} \\
x = \frac{5}{4} & a = 3 & b = 2 & p = 5 & \frac{6}{4}
\end{array}
$$

The Punchline

$$\frac{7}{5} < \sqrt{2} < \frac{6}{4} < \frac{5}{3} < \frac{4}{2} < \frac{3}{1}$$

Yes, this might be a coincidence. But it's a pretty pretty coincidence.

I Love p-adics

In case this isn't clear somehow, I love this weird math.

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I hope you enjoyed!