1 What Are p-adics?

This paper discusses the p-adic numbers, and the properties of the metric space under the p-adic metric. We'll complete this metric space and analyze some interesting properties of this space.

Motivation: We want another example of a metric space where we can analyze our definitions and see how our conclusions might differ, whether they confirm topological proofs that should hold over any arbitrary metric space or break our preconceived notions of metric spaces. The p-adics are a perfect space to do all of these as the p-adics are intrinsically related to $\mathbb{Q}$ but still provide very outlandish and at first, counter intuitive results.

To start, we'll explain what a p-adic number is. A p-adic number is a base p number where the significant digits are at the right of the number rather than the left.

Take for example, the 7-adics. Consider $\lim_{n\to\infty} 7^n$. This diverges with the traditional absolute value. However, we'll list out the first few values of 7^n in base 7 (From this point onward, if a number is not in base 10, we'll put a subscript to denote it's base).

$$7^1 = 10_7$$
$$7^2 = 100_7$$
$$7^3 = 1000_7$$
$$7^4 = 10000_7$$

We'll write some values suggestively.

$$7^5 = \ldots 000_7$$
$$7^6 = \ldots 000_7$$
$$7^7 = \ldots 000_7$$

Since the right most digits are the most significant, we say that $\lim_{n\to\infty} 7^n$ converges to $\ldots 000_7$ p-adically.

We can do this with any value of p and see that p^n converges to $\ldots 000_p$ p-adically.

Now, we formalize this.

Definition 1 Take any $x \in \mathbb{Q}$. Write out the prime factorization of x. Let $p \in \mathbb{N}$.

$x = a_1^{r_1} \cdots p^{r_i} \cdots a_k^{r_k}$ for some $i \in \mathbb{N}$.

Then the p-adic absolute value is: $|x|_p = \dfrac{1}{p^{r_i}}$

We'll investigate why $x \in \mathbb{Q}$ in a bit.

How does the p-adic[1] absolute value relate to leftmost digits being most important? Consider $p = 7$ again. Then, if there's a 7 in the prime factorization of x, then x_7 ends in a 0.

For example, $|7|_7 = \dfrac{1}{7}$. Also, $7 = 10_7$. See how the last digit is a 0?

[1] Shockingly, p-adic numbers have real world applications. For example, Monsky's Theorem, which states that "it is not possible to dissect a square into an odd number of triangles of equal area," requires the 2-adics to prove.

We'll run through a few more examples.

$$\begin{aligned}
|25|_5 &= \frac{1}{25}, 25 = 100_5 \\
|50|_5 &= \frac{1}{25}, 50 = 200_5 \\
|51|_{10} &= 1, 51 = 51_{10} \\
|1|_p &= 1, 1 = 1_p \\
|64|_2 &= \frac{1}{64}, 64 = 1000000_2 \\
|3005|_2 &= 1, 3005 = 101110111101_2 \\
|0|_p &= 0
\end{aligned}$$

Since there is no proper prime factorization of 0, it's p-adic absolute value is 0, since $p^r \neq 0$ for all $p, r \in \mathbb{N}$ (formally there's more going on here, but this is sufficient).

But wait! $0 \notin \mathbb{N}$. Well, we can extend prime factorizations to not just $\mathbb{Z}$ but also $\mathbb{Q}$!

We'll say that for $x \in \mathbb{Z}$, $|x|_p = |-x|_p$ since $-x$ and x have the same prime factorization.

If we allow negative exponents, we can write any number in $\mathbb{Q}$ as a unique prime factorization. For example, let $x = \frac{100}{99}$. We know that $x = 2^2 \cdot 3^{-2} \cdot 5^2 \cdot 11^{-1}$. Thus, $|x|_3 = 9$ while $|x|_2 = \frac{1}{4}$.

This definition is well defined since $x \in \mathbb{Q} \Leftrightarrow x = \frac{a}{b}, a \in \mathbb{Z}, b \in \mathbb{N}$. Since a, b both have prime factorizations, and are in lowest terms, this definition is well defined.

The formal language to describe the exponent of a prime is called the valuation of a number x. We denote this: $v_p(x)$.

If we use x from the previous example, $v_3(x) = 2$. Likewise, $v_3(x) = -2$.

We formally define this.

Definition 2 Take any $x \in \mathbb{Q}$. Write out the prime factorization of x. Let $p \in \mathbb{N}$.

$x = a_1^{r_1} \cdots p^{r_i} \cdots a_k^{r_k}$ for some $i \in \mathbb{N}$.

Then the valuation of x at prime p is: $v_p(x) = r_i$

Thus, we have an alternate way to define the absolute value:

Definition 3 $|x|_p = p^{-v_p(x)}$

But more importantly, p-adics are used in Number Theory!

While this absolute value seems useless, it gives us access to one of the weirdest intersections of Number Theory and Elementary Analysis. p-adic topology on $\mathbb{Q}_p$.

2 Discovering $\mathbb{Q}_p$

We'll now define the p-norm.

Definition 4 Let $x_1 \in \mathbb{Q}$ and p be prime.

Then, the p-norm is: $\|x_1\|_p = |x_1|_p$

We show that this is indeed a norm. [2]

[2]The definition of an absolute value is the same as a norm. Thus, we are also showing this is an absolute value.

Proposition 5 The p-norm is a norm.

Proof. We need to prove the p-adic absolute value obeys three properties: Positive definiteness, Homogeneity and Triangle inequality.

Start with Positive Definiteness.

By definition, $|x|_p = p^{-v_p(x)} \geq 0$ since $p \geq 2$.

If $x = 0$, then $0^n = 0$. Thus, $|x|_p = 0$.

If $x \neq 0$, we have two cases. If p is in the prime factorization of x, then $|x|_p = \frac{1}{p^{r_i}} > 0$. If p is not in the prime factorization, then $|x_p| = \frac{1}{p^0} = 1$.

Thus, $|x|_p = 0 \Leftrightarrow x = 0$ and $|x|_p > 0$ for all $x \in \mathbb{Q}$. Positive definiteness holds.

Homogeneity [3] in this case states that: $\forall x, c \in \mathbb{Q}, \|cx\|_p = \|c\|_p \|x\|_p$. Let $x, c \in \mathbb{Q}$ be given. we have:

$\|cx\|_p = |cx|_p = p^{-v_c(x) \cdot - v_p(x)} = p^{-v_c(x)} \cdot p^{-v_p(x)} = |c|_p |x|_p = \|c\|_p \|x\|_p$.

Finally the Triangle Inequality:

We'll show the following: $|x_1 + x_2|_p \leq \max\{|x_1|_p, |x_2|_p\}$. This property is called the Strong Triangle Inequality and by conseqeunce, we'll get the Triangle Inequality for free.

We'll first prove the Strong Triangle Inequality for integers. Let $x_1, x_2 \in \mathbb{Z}$. Then we'll show that the Strong Triangle Inequality will hold.

Note that since integers will only have non-negative exponents, we must have that: $|x_1 + x_2|_p \leq 1$. Let $|x_1 + x_2|_p$ be given so that $v_p(x) = i$. Then, $x_1 + x_2 \equiv 0 \mod p^i$ and $x_1 + x_2 \not\equiv 0 \mod p^{i+1}$.

Thus, either (a) $p^{i+1} \nmid x_1$ and $p^{i+1} | x_2$, (b) $p^{i+1} \nmid x_2$ and $p^{i+1} | x_1$ or (c) $p^{i+1} \nmid x_1, x_2$.

In case (a), $|x_1|_p \leq \dfrac{1}{p^i}$. In the case of (b), $|x_2|_p \leq \dfrac{1}{p^i}$. In the case of (c), $|x_1|_p \leq \dfrac{1}{p^i}$.

Now, prove the Strong Triangle Inequality for all rationals. Let $x_1, x_2 \in \mathbb{Q}$. Then, we can write $x_1 = \dfrac{a}{b}$ and $x_2 = \dfrac{c}{d}$ with $a, b \in \mathbb{Z}$ and $c, b \in \mathbb{N}$.

Then, we have:

$$\begin{aligned}
|x_1 + x_2|_p &= \left|\frac{a}{b} + \frac{c}{d}\right|_p \\
&= \left|\frac{ad + bc}{bd}\right|_p \\
&= \left|\frac{1}{bd}\right|_p \cdot |ad + bc|_p \\
&\leq \left|\frac{1}{bd}\right|_p \cdot \min\{|ad|_p, |bc|_p\} \\
&= \min\{\left|\frac{ad}{bd}\right|_p, \left|\frac{bc}{bd}\right|_p\} \\
&= \min\{\left|\frac{a}{b}\right|_p, \left|\frac{c}{d}\right|_p\} \\
&= \min\{x_1, x_2\}
\end{aligned}$$

Thus, the p-norm is indeed a norm.

[3] Theoretically, we can define the p-adics for any $p \in \mathbb{Z}$. However, it's only an absolute value and field when p is prime. If, for example, $p = 6$, then $\frac{1}{6} = |6|_6 = |2|_6 |3|_6 = 1 \cdot 1 = 1$. Clearly, contradiction.

$\square$

Since $|x|_p$ is a norm we will induce a metric (as proven in Eaxmple 1.3.2 in Notes.tex).

Definition 6 Let $x_1, x_2 \in \mathbb{Q}$. Then, we say the p-metric is: $d(x_1, x_2) = |x_1 - x_2|_p$.

Thus, d_p is a valid metric and $(\mathbb{Q}, d_p)$ is a metric space! [4] Consider our space, which currently is: $(\mathbb{Q}, d_p)$.

We need to do one more thing before we can analyze this space and some of its interesting properties. We have to complete the space. This completion is called $\mathbb{Q}_p$. We'll do this by Big List 41.

Let's start with how Big List 41 works when completing $\mathbb{Q}$ canonically.

Completeness states that every Cauchy sequence converges to a point within the set. However, we can construct a sequence in $\mathbb{Q}$ that is Cauchy but converges to a point outside of $\mathbb{Q}$. For example, $\sqrt{2}$. We take all the sequences that converge to the same point, and put them in an equivalence class based on the limit point. Then our space is the set of all equivalence classes. Doing this for every Cauchy sequence results in a completion of the space.

For $(\mathbb{Q}, d_5)$, an example of a Cauchy sequence that doesn't converge within $\mathbb{Q}$ is the following sequence:

$$(x_n) = \{5^2 \cdot \frac{31}{10}, 5^4 \cdot \frac{314}{100}, 5^6 \cdot \frac{3141}{1000}, 5^8 \cdot \frac{31414}{10000}, 5^{10} \cdot \frac{314159}{100000}, \ldots\}$$

This is a little scuffed[5] and there might be a better example, but this sequence is Cauchy and converges to $5^\infty \cdot \pi$. So while it diverges to infinity in our traditional absolute value, it converges 5-adicly. [6]

Formally, $\mathbb{Q}_p$ is the set of equivalence classes of Cauchy sequences, where each equivalence class contains all the sequences that converge to the same limit point.

i.e. Let (a_i) and (a_j) be Cauchy sequences that converge to a. Define $[a] = \{(x_n) \to a\}$. Then, $(a_i) \in [a]$ and $(a_j) \in [a]$.

Let $\epsilon > 0$. We say that if $|a_i - a_j|_p < \epsilon$ for all $i, j > N \in \mathbb{N}$, then these sequences are equivalent. Thus, they belong to the same equivalence class.

Definition 7 The set of all equivalence classes is $\mathbb{Q}_p$, and the p-adic norm of an equivalence class $[a]$ is $|lim_{i \to \inf}(a_i)|_p$ where $(a_i) \in [a]$.

Thus, the elements in our space will be the limit points of $\mathbb{Q}_p$. We need to define addition and multiplication in $\mathbb{Q}_p$ since we'll using these operators very frequently.

Define addition as follow:

Definition 8 Let $(a_n) \in [a]$ and $(b_n) \in [b]$. Then, $[a + b] = (a_n + b_n)$, where we add the terms in the sequence.

Clearly, $(a_n + b_n)$ converges since both are Cauchy.

Proposition 9 This definition is well defined.

Proof. Let $\epsilon > 0$. Take $(a'_n) \in [a]$ and $(b'_n) \in [b]$. Then, $|(a_n + b_n) - (a'_n + b'_n)|_p \leq \max\{|a_n - a'_n|_p, |b_n - b'_n|_p\} = \max\{|a - a|_p, |b - b|_p\} = 0$. $\square$

Define multiplication as follow:

[4] Since we have the strong triangle inequality, this is also considered an ultrametric space! Some of our conclusions later in the paper come directly from the Strong Triangle Inequality and hold over arbitrary ultrametric spaces.

[5] Formally defining Cauchy sequences that don't converge p-adically requires quadratic residues, properties of p-adics and copious amounts of modular arithmetic. It's outside of the scope of this paper.

[6] Interesting, if we multiply the ith term by 7^{-2i}, this sequence approaches 0 in the traditional absolute value but still converges in the 5-adics! Thus, we get convergent behaviour at both 0 and infinity. Likewise, we can get divergent behaviour at both of these extremes.

Definition 10 Let $(a_n) \in [a]$ and $(b_n) \in [b]$. Then, $[a \cdot b] = (a_n \cdot b_n)$, where we multiply the terms in the sequence.

Clearly, $(a_n \cdot b_n)$ converges since both are Cauchy.

Proposition 11 This definition is well defined.

Proof. Let $\epsilon > 0$. Take $(a'_n) \in [a]$ and $(b'_n) \in [b]$. Then, $|(a_n b_n) - (a'_n b'_n)|_p = |(a_n b_n) - (a'_n b_n) + (a'_n b_n) - (a'_n b'_n)|_p = |b_n(a_n - a'_n) + a'_n(b_n - b'_n)|_p \leq \max\{|b_n(a_n - a'_n)|_p, |a'_n(b_n - b'_n)|_p\} = \max\{b(a-a), a(b-b)\} = 0$. $\square$

Now we'll show that these two operations on $\mathbb{Q}_p$ form a field.

Proposition 12 $(Q_p, +, \cdot)$ is a field.

Proof. Let p be prime.

Note that associativity, distributivity and commutativity comes directly from how we define addition and multiplication. Since we add/multiply each term of the sequence and each term is from $\mathbb{Q}$, we must have that the sequences themselves are associative, distributive and commutative. However, we need to show closure and inverses.

Closure is immediate since every limit point is in $\mathbb{Q}_p$. Thus, we can never add or multiply two elements and result in a sequence that doesn't converge within the set.

We start by proving the existence of an additive inverse. Let (a_n) be a convergent sequence that converges to a with $[a] \in \mathbb{Q}_p$. Then, $(a_n) \in [a]$. We'll construct (b_n) where for each $i \in \mathbb{N}$, we have: $b_i = -a_i$. Thus, $a_i - b_i = 0$ and (b_n) converges as (a_n) converges. Let $(b_n) \to b$ so that $(b_n) \in [b]$. Then, the sequence $(a_n - b_n) = \{0, 0, 0, \ldots\} \to 0$. Since 0 is our additive identity, $[b]$ is the additive identity for $[a]$.

Proving that the multiplicative inverse exists is challenging[7], so we'll assume it to be true. Since we won't need multiplicative inverses in this paper, we'll take it as a given. Regardless, this still makes Q_p a ring. $\square$

Before we deal with the crux of the paper, I want to draw your attention to Ostrowski's theorem (1916): "Every non-trivial absolute value on the rational numbers $\mathbb{Q}$ is equivalent to either the usual real absoulte value or a p-adic absolute value." Note: The trivial absolute value is similar to the discrete metric, the following piecewise function:

$$|x|_0 = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

In this proof, they consider two absolute values to be equivalent if they induce the same topology. Thus, $\mathbb{Q}_p$ and $\mathbb{R}$ are the only two non-trivial completions of $\mathbb{Q}$.

3 p-adic Topology

The boring part first. $\mathbb{Q}_p$ is dense in itself since every space is dense in itself. It wouldn't make sense to compare $\mathbb{Q}_p$ with $\mathbb{R}$, so we'll take density as given. Then, by definition, $\mathbb{Q} \subset \mathbb{Q}_p$ is dense in X since $\overline{\mathbb{Q}} = \mathbb{Q}_p$ since we are completing $\mathbb{Q}$ with the p-adic metric. Thus, $\mathbb{Q}_p$ is seperable. I won't expand on density further.

Now, interesting stuff!

[7] This unfortunately comes down to the algebraic definition of p-adic rationals. The inverses of p-adic numbers have to do with modular arithmetic and infinite series. That's outside of the scope of this paper.

We need to first verify that we can use the same definition of a closed ball as we do in $\mathbb{R}$; Consider $B(x,r)$. Is $y \in \overline{B}(x,r) \Leftrightarrow |x-y|_p \leq r$? We'll prove this right away:

Proposition 13 $y \in \overline{B}(x,r) \Leftrightarrow |x-y|_p \leq r$

Proof. Let p be given in d_p. Let $0 < \varepsilon < \frac{r}{p}$.

We prove the forward direction: Assume $y \in \overline{B}(x,r)$ This means that y is a limit point of some sequence in $B(x,r)$. Let $(y_n) \in B(x,r)$ be the arbitrary sequence so that $(y_n) \to y$. In order for (y_n) to converge, we must have that: $|y_n - y|_p \leq \varepsilon$ for all $n > N$ where N is given.

By Strong Triangle Inequality, since $y_n \in B(x,r)$, we must have that: $|x-y|_p \leq \max\{|x-y_n|_p, |y_n-y|_p\} \leq \max\{r, \varepsilon\} = r$. Thus, $|x-y|_p \leq r$.

We continue by proving the reverse direction via contrapositive: $|x-y| > r \Rightarrow y \notin \bar{(B)}(x,r)$. Assume for the sake of contradiction that y is a limit point of $B(x,r)$. We'll use a similar proof as above.

Let $(y_n) \in B(x,r)$ be the arbitrary sequence so that $(y_n) \to y$. In order for (y_n) to converge, we must have that: $|y_n - y|_p \leq \varepsilon$ for all $n > N$ where N is given.

By Strong Triangle Inequality, we must have that: $|x-y|_p \leq \max\{|x-y_n|_p, |y_n-y|_p\} \leq \max\{r, \varepsilon\} = r$. Thus, $|x-y|_p \leq r$. But this leads to a contradiction as $|x-y|_p > r$. So our assumption must be false.

□

So the closed ball acts exactly as we expect it to. We continue analyzing closed/open sets.

Let's look at $B(0,r)$. This ball corresponds to all points x where $|x|_p < r$. Changing r should lead us to the following revelation: The ball doesn't act continuously.

To see this, consider $\mathbb{Q}_7$ and $B(0, \frac{1}{2})$. The elements in $B(0, \frac{1}{2})$ are all points $x \in \mathbb{Q}_7$ where $|x|_7 < \frac{1}{2}$. But since $|x|_7 = 7^l, \forall l \in \mathbb{Z}$, we can shrink the ball to have radius $r = \frac{1}{3}$ and it will be the same ball. Thus, given arbitrary $x \in \mathbb{Q}_p, B(x, \frac{1}{2}) = B(x, \frac{1}{3})$.

But this also shows that $B(x,r) = \overline{B}(x,r) \Leftrightarrow r \neq p^l$ for all $l \in \mathbb{Z}$.

The only time the closure of an open ball contains more elements is if $r = p^l$ for some $l \in \mathbb{Z}$ by contrapositive of our first statement. Let's prove this claim:

Proposition 14 $B(x,r) = \overline{B}(x,r) \Leftrightarrow r \neq p^l$ for all $l \in \mathbb{Z}$.

Proof. Let p be given in d_p and x be given in $\mathbb{Q}_p$.

We start by proving the forward direction by contrapositive: $r = p^l$ for some $l \in \mathbb{Z} \Rightarrow B(x,r) \neq \overline{B}(x,r)$. By definition, a point $y \in B(x,r)$ if $|x-y|_p < r = p^l$. Similarly, a point $y \in \overline{B}(x,r)$ if $|x-y|_p \leq r = p^l$. Take y so that $|x-y|_p = r = p^l$. Note that this is possible since $|\cdot|_p = p^n$ for all $n \in \mathbb{Z}$.

But notice that $y \in \overline{B}(x,r)$ but $y \notin B(x,r)$. Thus, $B(x,r) \neq \overline{B}(x,r)$ as $\overline{B}(x,r) \not\subseteq B(x,r)$.

Proceed with the reverse direction:

$r \neq p^l$ for all $l \in \mathbb{Z} \Rightarrow B(x,r) = \overline{B}(x,r)$. Note that since $|\cdot|_p = p^n$ for all $n \in \mathbb{Z}$. It's impossible that $|x-y|_p = r = p^l$ for all $y \in \mathbb{Q}_p$. Thus, since equality can never hold, we can reduce the definition of our closed ball: A point $y \in \overline{B}(x,r)$ if $|x-y|_p < r = p^l$. But this is precisely the definition for $B(x,r)$.

Since the sets are defined in the same way, we must have that: $B(x,r) = \overline{B}(x,r)$.

□

We'll do another small proof.

Proposition 15 $p^{l-1} \leq r_1, r_2 < p^l \Rightarrow B(x,r_1) = B(x,r_2)$

Proof. We'll show one subset inclusion and the other follows similarly, Take $y \in B(x, r_1)$. If $|x-y|_p < p^{l-1}$, then $|x-y|_p < r_2 \Rightarrow y \in B(x, r_2)$. If $|x-y|_p = p^{l-1}$, then $|x-y|_p \leq r_2 \Rightarrow y \in B(x, r_2)$. Note that if $|x-y|_p > p^{l-1}$, then $|x-y|_p \geq p^l$ which is a contradiction.

□

We get a very interesting property. Every ball is clopen! If given an open set, if $r = p^l$, we can reduce r and achieve a closed set. Conversely, if we have a closed set, if $r = p^l$ we can increase r and acheive an open set. Otherwise, r yields a clopen set.

This lines up with a very interesting fact that I'll assume without proof: A set is clopen if and only if the boundary of the set is empty. Indeed, the boundary of these clopen sets are empty!

While this might be very surprising, here's another shocking fact. Every point in a ball is the center of the ball.

Proposition 16 $|x-y|_p < r \Rightarrow B(x,r) = B(y,r)$

Proof. We'll do a double subset inclusion. We'll only include one subset inclusion as the other follows similarly.

Take $z \in B(x,r)$. Thus, $|x-z|_p < r$. Applying the Strong Triangle Inequality gives us: $|y-z|_p \leq \max\{|y-x|_p, |x-z|_p\} < r$. Thus, $z \in B(y,r)$. Since z is arbitrary, we must have that $B(x,r) \subseteq B(y,r)$.

Since the other subset inclusion follows similarly, we have that: $B(x,r) = B(y,r)$.

□

This is super unintuitive. And it all comes down to this Strong Triangle Inequality.[8]

Now, let's look at singletons. Let $\{x\} \in \mathbb{Q}_p$ be a singleton with arbitrary x. Singletons are closed since the only sequence in the singleton is: $\{x, x, x, \ldots\}$. Since the limit of this sequence is x, the limit point is in $\{x\}$.

However, we can show that singletons are not open using d_p.

Proposition 17 Singletons are not open in Q_p under d_p.

Proof. Let $\epsilon > 0$ and $p \in d_p$ be given. We'll show that $B(x,\epsilon) \not\subseteq \{x\}$. By Archimedian Principle, we know that there exists $N \in \mathbb{N}$ so that $\frac{1}{N} < \epsilon$. Take $l \in \mathbb{Z}$ so that $p^l < \frac{1}{N} < \epsilon$.

Then, $B(x, p^l) \subseteq B(x, \epsilon)$. Note that $|p^l|_p = \frac{1}{p^{-l}}$. Thus, by additive inverses, we have y so that $x - y = 0$. Then, let $z = y - p^{-l}$. Then, $|x-z|_p = |x - y + p^{-l}|_p = |p^{-l}|_p = p^l$. Thus, $z \in clB(x, p^l) \Rightarrow z \in B(x, \epsilon)$.

Since $x \neq z$, we must have that $B(x,\epsilon) \not\subseteq \{x\}$. □

We can use this fact to show that $\mathbb{Q}_p$ is extremally (not extremely) disconnected, which by definition states: A space is extremally disconnected if the closure of every open set is open. Since every clopen set's closure is open, all that matters is the singletons. But since singletons aren't open, this doesn't matter. Thus, $\mathbb{Q}_p$ is extremally disconnected.

Regarding connectedness, we can use 1.10.6 from the Notes which states that: X is disconnected $\Leftrightarrow U \subsetneq X$ with $U \neq \varnothing$ and U clopen, to conclude that $\mathbb{Q}_p$ isn't connected.

However, we will prove something even stronger using 1.10.6:

Proposition 18 $\mathbb{Q}_p$ is totally disconnected. Being totally disconnected means that the only connected sets are singletons.

[8] Since we never used the definition of the p-adics, this proof shows that any ultrametric space has this weird property.

Proof. Start by showing that singletons are connected. Take $\{x\}$ to be arbitrary. Since $|\{x\}| = 1$, then we can't construct A and B so that both are non-empty and disjoint, as $|A \cup B| = |A| + |B| = 2 \neq 1$. So singletons are connected.

Consider an arbitrary set $X \subseteq \mathbb{Q}_p$ where $|X| > 1$. Let the universal set in all cases be X.

If $X = \mathbb{Q}_p$, take $U = \overline{B}(0,1)$ and $V = \overline{B}(0,1)^c$. Since U is clopen, V is open. Then, $\mathbb{Q}_p = U \cup V$ with both U, V non-empty, open and disjoint.

If X is a clopen ball, say $B(x,r)$, then let $U = B(x, \frac{r}{p})$ where p is the same as in $\mathbb{Q}_p$. Then, $V = B(x, \frac{r}{p})^c \neq \varnothing$. Then, $X = U \cup V$ is disconnected as U being clopen implies that V is open. Since both are non-empty, X is disconnected.

If X consists of clopen balls, then choosing any clopen ball B yields: $X = B \cup B^c$. But since B is closed, B^c is open and both are not empty as X consists of clopen balls. Thus, X is disconnected.

If X is a union of singletons, we can always find some $\epsilon > 0$ so that two arbitrary points x, y have property: $y \notin B(x, \epsilon)$.

Thus, setting epsilon to be the infimum of epsilons for all points will make $B(x,\epsilon)$ clopen. If the infimum approaches 0, then choose another point. Note that this is possible since if every point had an infimum of 0 from every other point, X would have to be complete. Thus, X is either a clopen ball, a collection of clopen balls or $\mathbb{Q}_p$, which are all disconnected.

Then, $(B(x,\epsilon))$ being clopen makes $(B(x,\epsilon))^c$ open. Then, $X = (B(x,\epsilon)) \cup (B(x,\epsilon))^c$.

Thus, every possible construction of X where $|X| > 1$ is disconnected. So $\mathbb{Q}_p$ is totally disconnected.

□

Ok. Just one more thing left to do. Compactness.

We should see that just like $\mathbb{R}$, the whole space $\mathbb{Q}_p$ isn't compact. However, we can show that any ball is locally compact! We'll show that an arbitrary ball $B(x,r)$ is totally bounded. Since each ball is complete as a closed subset of a complete space is complete[9], by Big List 45, we have: "(X,d) is compact if and only if it is totally bounded and complete" will show that $\mathbb{Q}_p$ is locally compact for any sized ball.

We will construct the open covers directly so that our ball is locally compact. And yes, we're using open balls in $\mathbb{Q}_p$ to do this.

Proposition 19 Any finitely sized ball in $\mathbb{Q}_p$ is locally compact.

Proof. Let $0 < \epsilon < r$, p be given in $\mathbb{Q}_p$ and x given. Then consider $B(x,r)$. Let $l \in \mathbb{Z}$ be given so that $p^l < r \leq p^{l+1}$. Let $m \in \mathbb{Z}$ be given so that $p^m < \epsilon \leq p^{m+1}$. Then, we'll construct an open cover of balls with size ϵ of $B(x,r) = B(x,p^l)$. This will show that $B(x,r)$ is totally bounded.

The exact number of open balls we'll need is p^{l-m}. We'll define this recursively where we split an arbitrary $B(x,p^i)$ into p many $B(x_1,p^{i-1}), \ldots, B(x_p,p^{i-1})$ for $m < i \leq l$. We can keep applying this formula until $i = m$ and we cover all of $B(x,r)$.

In the case that $i = m$, then $B(x,\epsilon)$ covers $B(x,r)$ as $B(x,\epsilon) = B(x,p^i) = B(x,r)$ is open.

If $i \neq m$, then define $x_1 = x + p^i, x_2 = x + 2p^i, x_3 + 3p^i, \ldots, x_p = x + p \cdot p^i = p^{i+1}$. Then, $B(x,p^i) \subseteq B(x_1,p^{i-1}) \cup \cdots \cup B(x_p,p^{i-1})$.

We can see this since: If $y \in B(x,p^i)$, then $|x-y|_p < p^i$. We can rewrite this as: $x \equiv y \mod p^{-i}$. Then, we must have that: $x + kp^{-i} \equiv y \mod p^{-i+1}$ for some $k \in \{0,1,\ldots,p-1\}$. But then, $|x_k - y|_p < p^{i-1} \Rightarrow y \in B(x_k, p^{i-1})$.

Thus, we must have that $B(x,p^i) \subseteq B(x_1,p^{i-1}) \cup \cdots \cup B(x_p,p^{i-1})$. Since we can keep repeating this process and still end up with a finite number of open covers, $B(x,r)$ is totally bounded.

Thus, by Big List 45, $B(x,r)$ is compact for all $r > 0$ and by extension, $\mathbb{Q}_p$ is locally compact.

[9] This is true since every limit point of our closed subset is from $\mathbb{Q}_p$

□

By showing that $\mathbb{Q}_p$ is locally compact, by Big List 47, we see that $\mathbb{Q}_p$ is indeed, finite-dimensional.

You might be wondering, what if instead of completing $\mathbb{Q}$ with the p-adic metric, we completed $\mathbb{Z}$ with p-adic metric? Well, there are these things called the p-adic integers, denoted $\mathbb{Z}_p$. While the algebraic construction for this is complicated, we can define $Z_p = \overline{B}(0, 1)$. Intuitively, this makes sense, since only fractions have p-adic values greater than 1. So, let's just disregard $\mathbb{Z}_p$ since it's more interesting from an algebraic perspective.

So there you have it. Open balls are compact in the p-adics, nothing except singletons are connected, closure of open sets are still open, every ball has many centers, every ball is clopen, the jumps between different sized balls are discrete, and yet, the closure of an open ball is intuitive.

4 Conclusion

Regarding future directions and my next archive, because of how nicely p-adic topology works out, we can differentiate and integrate over $\mathbb{Q}_p$. However, because it's a heavy topic, many of the results are hard to prove in a short time. This means that my next paper on p-adic topology will be heavily based on resources outside of 257.

A few things before we finish the paper that you might find worth investigating:

1: Ultrametric spaces. What are some other ultrametric spaces? What are some interesting general properties of such spaces?

2: Product Topology. How does that generalize in Q_p^n? Is the max norm the only 'sensible' way to define product topology? Is there some way to generalize the euclidean norm or the p-norm? For example, how would you define the norm/metric on: $\mathbb{Q}_5 \times \mathbb{Q}_3$?

3: Interesting Properties. Are there other cool properties on $\mathbb{Q}_p$ that I failed to mention? There is at least one place in this paper where an implication could be made double-sided without much effort...

4: Algebraic Connections. The p-adics are very heavily based in Number Theory and Algebra. While most of the prerequisite knowledge of handling the p-adics from that perspective is not available to us, there are many interesting rabbit holes which might require study or a delay until a further time: What is $\mathbb{Z}_p$ from the algebraic perspective? What are alternate ways to construct $\mathbb{Q}_p$? Why is applying analysis to these algebraic objects so useful? Does applying analysis to these two spaces give us a better understanding of their structure?

5 Citations

”P-adic number,” Wikipedia, February 5, 2024,
https://en.wikipedia.org/wiki/P-adic_number.

Richeson, David. “What are p-adic numbers.” Division By Zero. Last modified November 24, 2008.
https://divisbyzero.com/2008/11/24/what-are-p-adic-numbers/.

Andrew Rich (2008) Leftist Numbers, The College Mathematics Journal,
39:5, 330-336, DOI: 10.1080/07468342.2008.11922313

”Monsky’s theorem,” Wikipedia, January 5, 2024,
https://en.wikipedia.org/wiki/Monsky%27s_theorem

Holloway, Paul James. “p-adic numbers.” Division By Zero. Last modified May, 2014.
https://people.math.carleton.ca/ cingalls/studentProjects/padicpaul.pdf

”Ostrowski’s theorem,” Wikipedia, December 11, 2023,
https://en.wikipedia.org/wiki/Ostrowski%27s_theorem

”Totally disconnected space,” Wikipedia, May 19, 2023,
https://en.wikipedia.org/wiki/Totally_disconnected_space

”Ultrametric space,” Wikipedia, July 16, 2023,
https://en.wikipedia.org/wiki/Ultrametric_space

Albeverio, Sergio, Andrei Khrennikov, and Vladimir M. Šelkovič. Theory of p-adic distributions: Linear and nonlinear models. Cambridge: Cambridge University Press, 2010.

Gouvêa, Fernando Q. *P-adic numbers: An introduction.* Cham, Switzerland: Springer Cham, 2020.