An Introduction to Mathematical Proofs

Bijections

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When? Whenever You Watch

We Have Returned!

We're back to the land of functions, and now, we'll formally show how to prove bijections!

We'll go through the topic as if we didn't cover it in the functions video. So don't be worried if you forgot what this word means!

Motivation

Functions are everywhere. So, we want some general ways to describe their properties.

Additionally, a really important thing is being able to invert functions.

I'll play a clip from the functions video to help us acheive both of these goals.

Injectivity

Let $f : A \to B$. Then, f is injective when elements from A uniquely map to elements in B. For example, if $3 \in B$, there's at most one element in A that maps to 3 (if nothing maps there, no issues).

How do we formally describe this? Consider the following definition:

Definition

Injectivity: We say f is injective if $\forall x_1, x_2 \in A$, we have that: $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$.
Intuitively, this means: For all pairs x_1, x_2 from our input set A, if the inputs aren't equal, then the outputs will never be equal.

We won't say one-to-one, but rather injective. If one-to-one makes more sense to you, use that for your own understanding.

Example

Example Let $f : \mathbb{R} \to \mathbb{R}$ be defined so that $f(x) = 2x + 2$ or, $x \mapsto 2x + 2$. Prove that this function is injective.

Proof. Let $x_1 \neq x_2$ be given from $\mathbb{R}$. Then, we have:

$$\begin{aligned} x_1 &\neq x_2 \\ 2x_1 &\neq 2x_2 \\ 2x_1 + 2 &\neq 2x_2 + 2 \\ f(x_1) &\neq f(x_2) \end{aligned}$$

Since x_1, x_2 were arbitrary, this holds for all $x_1 \neq x_2$. Thus, f is injective. $\square$

Note: We can't start from what we want to prove, thus we started from $x_1 \neq x_2$ rather than $f(x_1) \neq f(x_2)$.

Another Way To Prove

Example

Let $f : \mathbb{R} \to \mathbb{R}$ be defined so that $f(x) = 2x + 2$ or, $x \mapsto 2x + 2$. Prove that this function is injective. Let's prove this by contradiction.

Proof.

Assume for the sake of contradiction that there exists a pair of $x_1 \neq x_2$ from $\mathbb{R}$ so that $f(x_1) = f(x_2)$. We have:

$$f(x_1) = f(x_2)$$
$$2x_1 + 2 = 2x_2 + 2$$
$$x_1 = x_2$$

But this breaks our assumption that $x_1 \neq x_2$. Thus, we must have that $f(x_1) \neq f(x_2)$

$\square$

Finally, contrapositive is next.

Contrapositive

Example Let $f : \mathbb{R} \to \mathbb{R}$ be defined so that $f(x) = 2x + 2$ or, $x \mapsto 2x + 2$. Prove that this function is injective.

Proof. We'll prove the contrapositive. Namely, $\forall x_1, x_2 \in \mathbb{R}, f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.
Let $f(x_1) = f(x_2)$. Then we have:

$$f(x_1) = f(x_2)$$
$$2x_1 + 2 = 2x_2 + 2$$
$$x_1 = x_2$$

Since x_1, x_2 are arbitrary, we have shown that f is injective. $\square$

Notice, the contradiction and contrapositive are pretty similar, so stick with contrapositive as it's much clearer.

One Final Example

Example Let $f : (0, \infty) \to \mathbb{R}$ be given by $f(x) = e^x$. Show that f is injective.

Proof. We'll prove this by contrapositive. Let $x_1, x_2 \in (0, \infty)$. Then, we'll show that if $f(x_1) = f(x_2)$, then $x_1 = x_2$. We have:

$$\begin{aligned} f(x_1) &= f(x_2) \\ e^{x_1} &= e^{x_2} \\ ln(e^{x_1}) &= ln(e^{x_2}) \\ x_1 &= x_2 \end{aligned}$$

Thus, f is injective. $\square$

But This Isn’t The Whole Story

We still need one more property to invert functions properly.

Yes, I’m recycling content: Here’s another clip.

Surjectivity

Let $f : A \to B$. Then, f is surjective when elements from A map to every element in B. Or, the codomain of f is the range of f.

How do we formally describe this? Consider the following definition:

Definition

Surjectivity: We say f is surjective if $\forall y \in B, \exists x \in A$ such that $f(x) = y$.
Intuitively, this means: For every element y from our output set B, we can find some input element $x \in A$ so that $f(x) = y$.

We won't say onto, but rather surjective. If onto makes more sense to you, use that for your own understanding.

Example

Example Let $f : \mathbb{R} \to \mathbb{R}$ be defined so that $f(x) = 2x + 2$ or, $x \mapsto 2x + 2$. Prove that this function is surjective.

Proof. Let $y \in \mathbb{R}$ be an arbitrary element in our codomain. We claim that choosing $x = \frac{1}{2}y - 1$ shows that $f(x) = y$. To see this, consider the following:

$$\begin{aligned} f(x) &= 2x + 2 \\ &= 2(\frac{1}{2}y - 1) + 2 \\ &= y - 2 + 2 \\ &= y \end{aligned}$$

Since $f(x) = y$ for every value of y, our function is surjective. $\square$

Finding x

Unfortunately, there's no contrapositive or 'other' way to prove this. However, finding our value of x might be challenging...

In our rough work, once we're convinced a function is surjective, assume it's invertible. Then, invert the function and find our value of x.

Note: if the function isn't injective, then we'll get many possible values of x. We can choose any one of the x values.

A Tough Example

Example

Show that $f : \mathbb{R} \to [-0.25, \infty)$ given by $f(x) = x^2 + x$ is surjective.

Let's start with our rough work: First, we'll convince ourselves this function is surjective.

We can use calculus to see that both $x = \frac{1}{2}$ is our critical point.

Since a smaller x value and a larger x value is larger than at our critical point, the minimum is acheived at $f(-0.5) = -0.25$.

Since the codomain is above or at our minimum value and the function is increasing, we must have that every point is reached in our codomain.

More Rough Work

While the 'proof' on the previous slide somewhat rigorous, some people don't want to see you succeed, so we'll have to prove surjectivity formally through definition without using calculus.

Now, we'll assume that $y = x^2 + x$ is invertible for the sake of our rough work. Flipping x and y, we get: $x = y^2 + y \Rightarrow y^2 + y - x = 0$

Now, we'll have to use the quadratic formula. Since this yields two solutions, we can choose either. We have: $y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

In our case, choose $a = 1, b = 1, c = -x$. Then, $y = \frac{-1 \pm \sqrt{1+4x}}{2}$.

Uhhhh...

You'll never get something this annoying, but the process is as follows. Usually, the process won't require this many steps. Many logarithms or linear function.

Now, let's prove our surjectivity claim:

Proof

Proof. Let $y \in [-0.25, \infty)$. Then, we claim that $x = \frac{-1+\sqrt{1+4y}}{2}$ works. Note, in our rough work, we swapped x and y so here we swap it back and choose one of the solutions. We have:

$$
\begin{aligned}
f(x) &= x^2 + x \\
&= \left(\frac{-1+\sqrt{1+4y}}{2}\right)^2 + \frac{-1+\sqrt{1+4y}}{2} \\
&= \frac{1 - 2\sqrt{1+4y} + 1 + 4y - 2 + 2\sqrt{1+4y}}{4} \\
&= \frac{4y}{4} \\
&= y
\end{aligned}
$$

Thus, f is surjective. $\square$

Note

Once again, usually surjective proofs won't be this bad. But the process is the same. If you can't find the proper value of x, follow these steps. Ensure it's surjective, invert, choose one of the solutions and prove.

Last Step

More recycled content headed your way: Now we can invert functions.

Bijectivity

Let $f : A \to B$. Then,

Definition

Bijectivity: We say f is bijective when f is both injective and surjective.

We call f^{-1} the inverse function, where $f^{-1} : B \to A$ so that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

Intuitively, this means: For every element y from our output set B, can be mapped directly to some input element $x \in A$ so that $f(y) = x$ and this mapping is a function.

People call bijective functions both one-to-one and onto, rather than coming up with a new word. As always, we'll stick to bijective.

Indeed...

We've already shown that the function $f : \mathbb{R} \to \mathbb{R}, f(x) = 2x + 2$ is already a bijective function since we showed it's both injective and surjective.

Interestingly, the inverse function will always be how you prove surjectivity, but with the x and y flipped. In this case, $f^{-1} = \frac{1}{2}x - 1$.

Now, A Barrage

Decide whether the following functions are injective, surjective, both or neither.

Example $f : \mathbb{N} \to \mathbb{N}, f(x) = x + 1$

Example $f : \mathbb{Z} \to \mathbb{Z}, f(x) = x + 1$

Example $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$

Example $f : \mathbb{Z} \to \mathbb{N}, f(x) = |x| + 1$

Answers!

Example $f : \mathbb{N} \to \mathbb{N}, f(x) = x + 1$ is only injective. Nothing maps to 1.

Example $f : \mathbb{Z} \to \mathbb{Z}, f(x) = x + 1$ is bijective. Prove it!

Example $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$ is neither. Nothing maps to -1, and two values map to 1

Example $f : \mathbb{Z} \to \mathbb{N}, f(x) = |x| + 1$ is surjective. Assume you can invert the function and try to find a valid value for x.

Proof Sketch

For 1 and 2, $x_1 \neq x_2 \Rightarrow x_1 + 1 \neq x_2 + 1 \Rightarrow f(x_1) \neq f(x_2)$

For surjectivity in 2 and 4, choose $x = y - 1$.