An Introduction to Mathematical Proofs

Cardinality

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When? Whenever You Watch

Formally, Cardinality

If $A = \{1, 5, 8\}$, then $|A| = 3$. We say that the cardinality of A is 3.

The reason we don't use the word 'size' is because we allow infinity to be a valid 'size.' At least, that's what makes the most sense to me (yes, I was never told either, and I may be wrong, but at least I'm trying to make it make sense.)

Countability

Any finite set will have finite cardinality. But what about infinite sets?

Let's look at the simplest infinite set, $\mathbb{N}$.

We say that $\mathbb{N}$ has countable cardinality by definition, or, there are countably many elements in $\mathbb{N}$.

The reason we say $\mathbb{N}$ has countable cardinality is because we can 'count' or list or enumerate or find every element in $\mathbb{N}$. We can devise an algorithm to find every element, despite it being an infinite set.

Intuition

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The algorithm in this case is counting. $1, 2, 3$, and so on.

I hate how the word 'countable' isn't intuitive. I much prefer words like enumerable or iterable or listable. These make sense. Despite the set being infinite in size, we can devise an algorithm to enumerate or iterate or list every element.

Even Numbers

Let $E = \{2, 4, 6, \ldots\} = \{2n : n \in \mathbb{N}\}$.

What's $|E|$?

While $E \subsetneq \mathbb{N}$, we can devise an algorithm to list every even number and E is infinite. Thus, $|E| = |\mathbb{N}|$. Just add two starting at 2.

Formal Proof

But this isn't a proof. How could we prove that $|E| = |\mathbb{N}|$?

Well, if $\mathbb{N}$ is countable/enumerable/iterable/listable and so is E, then we should be able to come up with a way to match or pair even numbers and naturals together.

Wait, this just sounds like a bijection! We literally match every even number with every natural number!

Bijection Proof

We'll show that $f : \mathbb{N} \to E$ given by $f(x) = 2x$ is a bijection. Note, you can also do the function from $E \to \mathbb{N}$, the inverse, since it's also a bijection.

Proof. Start with injectivity. We'll use contrapositive. Let $f(x_1) = f(x_2)$. We want to show that $x_1 = x_2$. We have:

$$f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$$

Now, surjectivity. Let $y \in E$ be an arbitrary even number. Then, let $x = \frac{1}{2}y$. Since y is even, $x \in \mathbb{N}$. Now, we have: $f(x) = 2x = y$ □

Here's The Cool Part

If we instead set $E = \{0, 2, 4, \ldots\}$, we still have that $|E| = |\mathbb{N}|$.

Essentially, infinity is so powerful, that adding or removing finitely many things has no impact.

Injectivity, Surjectivity and Cardinality

Indeed, our logic from last video still makes sense. If $|A| \leq |B|$, we can find an injective function from A to B. Likewise, if $|A| \geq |B|$, we can find a surjective function from A to B.

We let countable cardinality be greater than any finite cardinality. Then, this conclusion still holds. We can always find an injective function from finitely many objects to countably many objects and find a surjective function from countably many objects to finitely many objects.

And the reverse is true, just like bijections. If $f : A \to B$ can never be surjective, then $|A| < |B|$. Thus, f can inform us on the cardinality of A and B.

Infinite PHP Part 1

Consider the set $A = \{1, 3, 6, 10, 15, 21, 28, 36, 45, \ldots\}$. What can we say about the ending digits of the elements in A without calculations?

A has countably many elements as $|A| = |\mathbb{N}|$. Prove this on your own! But there are 10 possible ending digits. Can we extend PHP to come up with an interesting conclusion?

We can. Countably many elements from A will share their ending digit.

Infinite PHP Part 1 Second Half

"Countably many elements from A will share their ending digit." Why is this true?

If we have finitely many boxes but infinite pigeons, we can't have a finite amount of pigeons in every box! Or else, we wouldn't have had infinitely many pigeons!

So one box must have infinite pigeons while the rest could or could not have infinite pigeons.

And here, by infinite, we mean countably many pigeons.

Intuition

Imagine we have one box and infinitely many pigeons. Well, all the pigeons must go to that one box.

What about two boxes? If both boxes had finitely many pigeons, then where do the rest of the pigeons go?

And we can keep extending this logic to every natural number, or every finite number.

A Slightly Harder Example

Okay, no more pigeons. Show that $|\mathbb{Z}| = |\mathbb{N}|$.

This is equivalent to finding a function $f : \mathbb{N} \to \mathbb{Z}$ where f is bijective.

First, we think about how we would list out all elements in $\mathbb{Z}$.

Why $\mathbb{Z}$ Is Countable

We usually write out $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. But how would we list every element?

We can't count from 1 to ∞ and then turn around when we reach ∞ because we'll never reach ∞. Or, this method can't list out every element in $\mathbb{Z}$ because we're stuck going forward.

So, what if we zig-zag? For example, $0, 1, -1, 2, -2, 3, -3, \ldots$? We'll reach every number and we don't get this issue of needing to turn around. Essentially, since the ... goes only in one direction just like $\mathbb{N}$, we're fine!

Mapping $\mathbb{N}$ to $\mathbb{Z}$

We just said $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$.
Likewise, $\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, \ldots\}$

So how about the following map:
$f(1) = 0, f(2) = 1, f(3) = -1, f(4) = 2,$
$f(5) = -2, f(6) = 3, f(7) = -3, \ldots?$

And this works. Now, to formally prove this, we need to define $f(x)$. While you could argue this without a formal rule for $f(x)$, it's hard to do and not too rigorous.

Proving $|\mathbb{Z}| = |\mathbb{N}|$

Let $f : \mathbb{N} \to \mathbb{Z}$ given by

$$f(x) = \begin{cases} 0 & \text{if } x = 1 \\ \frac{x}{2} & \text{if x is even} \\ -\frac{x-1}{2} & \text{if x is odd and } x \neq 1 \end{cases}$$

We'll show that f is bijective.

Injectivity

Proof. Start with injectivity. Let $x_1 \neq x_2$. We have three cases.

If $x_1 = 1$, then $f(x_1) = 0$. If x_2 is even, then $f(x_2) \neq 0$ as we divide by two. If x_2 is odd but not equal 1, then $f(x_2) = -\frac{x-1}{2} < 0$.

If x_1 is even, then $f(x_1) > 0$. Thus, x_2 must be even. But then: $x_1 \neq x_2 \Rightarrow \frac{x_1}{2} \neq \frac{x_2}{2} \Rightarrow f(x_1) \neq f(x_2)$

If x_1 is odd and not equal 1, then $f(x_1) < 0$. Thus, x_2 must also be odd and not equal 1. Then,

$x_1 \neq x_2 \Rightarrow -(x_1 - 1) \neq 1(x_2 - 1) \Rightarrow -\frac{x_1-1}{2} \neq -\frac{x_2-1}{2} \Rightarrow f(x_1) \neq f(x_2)$

In all cases, f is injective.

Surjectivity

Proof. Continued.

We'll show f is surjective. Let $y \in \mathbb{Z}$. We have three cases.

If $y = 0$, then $x = 1$ works as $f(x) = f(1) = 0 = y$.

If y is positive, then $x = 2y$ works. Note that x is even and a natural number. Then,
$f(x) = f(2y) = \frac{1}{2}(2y) = y$.

If y is negative, then $x = -2y - 1$ works. Note that x is odd and since $2y < -1$, we have that $-2y > 1$. Thus $x = -2y - 1 > 0$ so x is a natural number. We have:
$f(x) = f(-2y - 1) = -\frac{(-2y-1)+1}{2} = y$.

Thus, f is surjective and therefore, bijective. So,
$|\mathbb{N}| = |\mathbb{Z}|$.

Notes

1. When working with a piecewise function, the proofs have cases. We need to consider all cases or all possible definitions of f.

2. You might have noticed that in my surjectivity proofs, I show that x is in our domain. We've never had to deal with this before since the domain and codomain so far have always been $\mathbb{R}$.

I'm being extra careful with my proofs. You should be as careful as possible when doing proofs, granted you have time.