Answers are found at the bottom of the worksheet on a seperate page.

Question 1: Let P and Q be mathematical statements. First, express the following list in logical symbols (1 not included) and determine which of following statements are logically equivalent to each other.

1. $P \implies Q$.
2. The converse of $P \implies Q$.
3. The contrapositive of $P \implies Q$.
4. The negation of $P \implies Q$.
5. The converse of the contrapositve of $P \implies Q$.
6. The contrapositive of the converse of $P \implies Q$.
7. The converse of the converse of ... [2022 times] ... of $P \implies Q$.

For fun, consider (7) again. Can you generalize how this behaviour would change if it were a different number like 3031? What conclusions can you come up with?

Question 2: Prove by contradiction: If $a_1, a_2, \ldots, a_{2024} \in \mathbb{N}$ and $a_1 + a_2 + \ldots + a_{2024} \leq 2026$ then $(\forall 1 \leq i \leq 2024)[a_i \leq 3]$.

Hint: Try to understand what is being asked by unravelling the symbols, and then you'll see that the question isn't too hard.

Question 3: It is common for beginners to overuse proof by contradiction when it isn't needed. This can make the argument hard to understand.

Rewrite the following proof so that it is a direct proof, and not a proof by contradiction.

Let $x \in \mathbb{N}$. If $x > 1$, then $x^3 + 1$ is composite.

Proof. Assume that $x > 1$ and for a contradiction assume that $x^3 + 1$ is not composite. Since $x > 1$ we must have $x^3 + 1 > 1$. This means that $x^3 + 1$ is prime.

You can see that $(x + 1)(x^2 - x + 1) = x^3 + 1$, and both factors are integers. Since $x > 1$ we see that $1 < x + 1 < x^3 + 1$. Therefore $x^3 + 1$ is not prime. This contradicts that $x^3 + 1$ is prime. $\Rightarrow\Leftarrow$. $\square$

Question 4: For each of the following statements decide which proof technique (direct, contrapositive, contradiction) you should try first. Then, prove them.

1. Let $x \in \mathbb{Z}$. If 3 doesn't divide x, or, 3 isn't a factor of x, then 6 does not divide x, or, 6 isn't a factor of x either.
2. Let $m, n \in \mathbb{Z}$. If $m^2 + n^2$ is divisible by 4, or, 4 is a factor of $m^2 + n^2$, then both m and n are even numbers.
3. There are no rational solutions to $x + x^3 = 1$.

Hint: (3) is tough. There will come a point when you'll have to consider odds/evens. Also, (1) and (2) require you to understand factorization or divisibility. Since we didn't cover these topics formally, it's understandable if these questions are tough.

Question 5: Consider the following definition. I want you to write it in symbols only. Then, write the negation of the statement in both words and symbols.

Definition 1 Let c be a real number. Let $f : \mathbb{R} \to \mathbb{R}$. We say that the $lim_{x \to c} f(x)$ is equal to L when: For all values of ϵ greater than 0, we have that for any real valued x, we can find some δ that is also greater than zero so that the following implication is true: if $|x - c| < \delta$, then we have that: $|f(x) - L| < \epsilon$

Hint: You can start the symbolic representation as follows:

"Let $c \in \mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$. We say that the $lim_{x \to c} f(x)$ is equal to $L \in \mathbb{R}$ when:"

The reason is because this is hard to symbolify. What really matters is what proceeds this.

Remark: Logic is annoying to deal with and hard to learn. Fortunately, we're done with all the logic and we can proceed.

I know a math professor who doesn't enjoy logic either, so when he teaches this course, he tries to get through it as fast as possible.

Answer 1:

1. $P \implies Q$.
2. The converse of $P \implies Q$: $Q \Rightarrow P$
3. The contrapositive of $P \implies Q$: $\neg Q \Rightarrow \neg P$. This is logically equivalent to (1).
4. The negation of $P \implies Q$: $P \not\Rightarrow Q = P \wedge \neg Q$.
5. The converse of the contrapositve of $P \implies Q$: The converse of $\neg Q \Rightarrow \neg P$: $\neg P \Rightarrow \neg Q$. This is logically equivalent to (2).
6. The contrapositive of the converse of $P \implies Q$: The contrapositive of $Q \Rightarrow P$:$\neg P \Rightarrow \neg Q$. This is logically equivalent to (2) and (5).
7. The converse of the converse of ...[2022 times] ...of $P \implies Q$: Noticing that two converses 'cancel' or return the statement back to the original, Thus, doing the converse an even amount of time leaves the statement unchanged. In this case, this is logically equivalent to (1).

Answer 2:

Proof. Assume for the sake of contradiction that we can find some $a_i > 3$ but still have that $a_1 + a_2 = \cdots + a_{2024} \leq 2026$. Since every term must be at least one, we can remove a term and subtract one from the righthand side. We'll remove every term except a_i. we have that:

$$\begin{aligned} a_1 + a_2 + \cdots a_{2024} &\leq 2026 \\ a_2 + a_3 + \cdots a_{2024} &\leq 2025 \\ a_3 + a_4 + \cdots a_{2024} &\leq 2024 \\ a_i &\leq 3 \end{aligned}$$

But how can $a_i > 3$ and $a_i \leq 3$ at the same time? Contradiction. □

Answer 3:

Proof. Assume that $x > 1$.

You can see that $(x+1)(x^2 - x + 1) = x^3 + 1$, and both factors are integers. Since $x > 1$ we see that $1 < x + 1 < x^3 + 1$. Therefore $x^3 + 1$ is composite.

□

Answer 4:

1. Contrapositive. It's better to deal with existence of something rather than non-existence of something.
2. Contrapositive. We're comfortable dealing with odds and evens, so that's why we want to deal with them.
3. Contradiction. If we assume that a solution exists, we can use the definition of a rational, which we wouldn't be able to use otherwise.

Proof for (1):

Proof. We'll prove the contrapositive. Namely, if 6 is a factor of x, then so is 3.

If a is a factor of n, and b is a factor of a, then b is a factor of n. Since $6 = 3 \cdot 2$, by using properties of factorization, 3 is a factor of x.

□

Proof for (2):

Let $m, n \in \mathbb{Z}$. If $m^2 + n^2$ is divisible by 4, or, 4 is a factor of $m^2 + n^2$, then both m and n are even numbers.

Proof. We'll prove the contrapositive. Namely, if either m and n are odd, then 4 isn't a factor of m^2+n^2.

We'll assume without loss of generality (meaning, our assumption won't affect the truth of our proof) that m is odd. Then, m^2 is odd.

If n is even, then n^2 is even. But Odd + Even = Odd, so $n^2 + m^2$ can't be divided by 4 or have 4 as a factor.

If n is odd, then $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$. The remainder of n^2 when divided by 4 is 1. We can see that the remainder of m^2 will also be 1 when divided by 4.

Thus, $n^2 + m^2$ will have a remainder of 2 when divided by 4, thus it can't be divided by 4 cleanly, so 4 can't be a factor. □

Proof for (3):

Proof. Assume for the sake of contradiction that $x + x^3 = 1$ has a rational solution. Thus, we can write $x = \frac{a}{b}$ which is in lowest terms. We have:

$$\begin{aligned} x + x^3 &= 1 \\ \frac{a}{b} = \frac{a^3}{b^3} &= 1 \\ \frac{ab^2 + a^3}{b^3} &= 1 \\ ab^2 + a^3 &= b^3 \end{aligned}$$

Now, we analyse this last equation via cases for when $a.b$ are odd or even.

Case 1: Both a, b are even. Then, we could simplify $\frac{a}{b}$ by dividing numerator and denominator by two, but we assumed $\frac{a}{b}$ was in lowest terms. So we arrive at a contradiction.

Case 2: Both a, b are odd. Then, ab^2 is odd, a^3 is odd and b^3 is odd. But then, Odd + Odd = Odd? Contradiction.

Case 3: b even and a odd. Then, ab^2 is even, a^3 is odd and b^3 is even. But then, Even + Odd = Even? Contradiction.

Case 4: b odd and a even. Then, ab^2 is even, a^3 is even and b^3 is odd. But then, Even + Even = Odd? Contradiction.

In all cases, we arrive at a contradiction. Thus, we can never have that $ab^2 + a^3 = b^3$ and by reversing our steps, we could never have that $\frac{a}{b} + \frac{a^3}{b^3} = 1$. Thus, $x + x^3 + 1$ has no rational solutions. □

This last proof is tough since we have to realize we need to do a parity (even/odd) check *and* use the definition of rationals. Yeah, it took me some time to solve it as well, so don't feel bad if you didn't get it.

Answer 5: The symbolic representation:

Let $c \in \mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$. We say that the $lim_{x \to c} f(x)$ is equal to $L \in \mathbb{R}$ when:

$$\forall \epsilon > 0, \forall x \in \mathbb{R}, \exists \delta > 0, |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon.$$

We'll omit the English version of the negation and show the symbolic one. The reason I wanted you to write the English version is to understand the negation before you attempt to symbolify it.

Negation: Let $c \in \mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$. We say that the $lim_{x \to c} f(x)$ is not equal to $L \in \mathbb{R}$ when:

$$\exists \epsilon > 0, \exists x \in \mathbb{R}, \forall \delta > 0, |x - c| < \delta \wedge |f(x) - L| \geq \epsilon.$$

This is the formal definition for a limit to be defined at a point c and equal to L.

For an explanation of this, you can visit this video here

The whole YouTube channel linked above is a valuable resource since it's a series on calculus with proofs. Very interesting and worth watching.