An Introduction to Mathematical Proofs

Evens and Odds, Primes and Composites

Who? Fahad Hossaini

When? Whenever You Watch

Factors

When we can divide a positive non-decimal number x by y and have no fractions leftover, we call y a factor of x. Then, x and 1 are always factors of x.

Example $15 = 3 \cdot 5$. So, 3 and 5 are factors of 15

Example $56 = 7 \cdot 8 = 2 \cdot 28 = 4 \cdot 14$. Even if we only had the first part, we'd see that 7 and 8 are factors. But since 2 and 4 are factors of 8, 2 and 4 would also be factors of 56. 28 and 14 are also factors.

Evens and Odds - Formalization

Is 3 even?

What about 3.5?

Okay, what about -4?

Essentially, if we can divide a non-decimal number by two, it's even. Otherwise, it's odd!

Divisible By Two

Definition Even numbers can be written in the form: 2k, where k is another smaller number.

$14 = 2 \cdot 7$. Here, $k = 7$, so 14 is even. For -42, $k = -21$.

For odd numbers, we just add or subtract one. We'll go with adding one.

Definition Odd numbers can be written in the form: $2k + 1$, where k is another smaller number.

$29 = (2 \cdot 14) + 1$. Here, $k = 14$. For -35, $k = -18$.

Proofs With Evens and Odds

Now we have a formal definition of what it means for a number to be even or odd, so we can prove facts about evens and odds.

Let's prove this: Even $\cdot$ Even $=$ Even

Strategy One: Brute Force

Proof.

An even number ends in either 0,2,4,6 or 8. Thus, we can create a table to see what happens when we multiply the last digits of even numbers:

0	0	0
0	2	0
$\vdots$	$\vdots$	$\vdots$
8	6	8
8	8	6

Thus, since the result ends in 0,2,4,6 or 8, we have that Even $\cdot$ Even $=$ Even. $\square$

'Issues': We need to prove that even numbers ends in 0,2,4,6 or 8. Additionally, it doesn't feel 'clever,' whatever the hell that means.

Strategy Two: Definition Unwinding

Proof.

Take two arbitrary/random even numbers x and y. By definition, let $x = 2k$ for some smaller k (remember, this is by definition, so it doesn't matter what k is. All that matters is that k exists so that x is even.) and let $y = 2l$ for some smaller l.

Then, we have:

$$\begin{aligned} x \cdot y &= 2k \cdot 2l \\ &= 2(2kl) \end{aligned}$$

Since $2kl$ is a non-decimal number, and this is the form an even number takes, we must have that $x \cdot y$ is even. Since x and y were arbitrary or random choices, this must work for every even number. □

Exercise!

Remember, adding/subtracting and multiplying non-decimal numbers keep them as non-decimal numbers. You can use this fact without proof!

Now, show that Odd $+$ Odd $=$ Even, or Odd $\cdot$ Even $=$ Even.

Which Proof Technique?

We have two objectives when writing proofs.

1: Understand why the proof is true. We want to convince ourselves that a proof is true.

2: Communicate our proof to someone else. Either to validate our claims, or show that we understand the proof, as explaining a proof proves that you understand it.

Experiment and play around with different proof techniques, strategies and ideas. What works well for me might not work well for you and vice versa.

Example: Odd $\cdot$ Odd $=$ Odd

Proof. Let x, y be arbitrary odd numbers. Namely, let $x = 2k + 1$ and let $y = 2l + 1$ for suitable k and l. Then, we multiply them together and manipulate the result to get the following:

$$\begin{aligned} x \cdot y &= (2k+1)(2l+1) \\ &= 4kl + 2k + 2l + 1 \\ &= 2(2kl + k + l) + 1 \end{aligned}$$

If we let $2kl + k + l = m$, then we get: $2m + 1$. Since this is the form of an odd number, then $2(2kl + k + l) + 1 = x \cdot y$ is an odd number. Since x and y were arbitrary, any odd number multiplied with another odd number must be odd. $\square$

Primes

Definition We say a non-decimal number p is prime if: $p \neq ab$ for any a, b that meet the criteria: $1 < a \leq b < p$.

Example: 1001 is not prime since $1001 = 13 \cdot 77$.

Example: 101 is prime since we can't write it in terms of smaller numbers being multiplied.

Definition A number that isn't prime is called composite. Or, $p = ab$ for some a, b that meet the criteria: $1 < a \leq b < p$.

$x^2 - 1$ is Composite when $x \geq 3$

Proof. Try First! $\square$

$x^2 - 1$ is Composite when $x \geq 3$

Proof. Use the following algebraic identity:

$$(x+y)(x-y) = x^2 - y^2$$

Then, when $y = 1$, we have that:

$$x^2 - 1^2 = (x+1)(x-1)$$

Then, we see that $1 < (x-1) \leq (x+1) < x^2$, so by definition, $x^2 - 1$ is composite. $\square$