Answers are found at the bottom of the worksheet on a seperate page.

Question 1: Express the following statements in English in a way that someone in grade 8 would understand. Are the following statements true or false? Justify your answer with a proof.

1. $(\exists x \in \mathbb{Z})[x^3 > 10]$
2. $(\forall x \in \mathbb{R})[x > 0 \lor x < 0]$.
3. $(\forall x \in \mathbb{R})[x = -2 \implies x^2 = 4]$
4. $(\forall x \in \mathbb{R})[x^2 = 4 \implies x = -2]$
5. $(\forall x \in \mathbb{R})[x^2 = 4 \iff (x = -2 \lor x = 2)]$

Question 2: This question helps distinguish between the order of quantifiers. The last question is "easy" if you understand quantifiers, and impossible otherwise.

Express the following statements in English in a way that someone in grade 8 would understand. Are the following statements true or false? Justify your answer with a proof.

1. $(\exists x \in \mathbb{N})(\exists y \in \mathbb{N})[x = -y]$
2. $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})[x < y \implies x^2 < y^2]$.
3. $(\forall x \in \mathbb{N})(\exists y \in \mathbb{Z})[x + y = 0]$
4. $(\exists y \in \mathbb{Z})(\forall x \in \mathbb{N})[x + y = 0]$
5. $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})(\forall z \in \mathbb{R})(\exists a \in \mathbb{R})[a + x = y + z]$

Question 3: The following activities will help you understand "vacuous truth".

1. Is the following true: "Every vowel in the word Toronto is an o.". How would you prove it?
2. Is the following true: "Every consonant in the word Toronto is a 't'.". How would you prove it?
3. Is the following true: "Every Greek letter in the word Toronto is a Σ."? How would you prove it?
4. Is the following true: "If x is a consonant in the word Toronto, then x is a 't'." How would you prove it?
5. Is the following true: "If x is a Greek letter in the word Toronto, then x is a Σ."? How would you prove it?
6. Is the following true: $(\forall x \in \mathbb{R})[x^2 < 0 \implies x > 0]$?

Question 4: Prove that $x + y \geq x$ if and only if $y \geq 0$. Unfortunately, these questions are essentially repeated from before, but don't worry, from this worksheet onwards, we can ask more interesting questions.

Remark: It's quite surprising how expansive implications are. Here's a somewhat difficult example to illustrate this point:

$\forall n \in \mathbb{N} \setminus 2, n$ is even $\Rightarrow \exists x, y \in \mathbb{N}$ such that $x + y = n \land$ x is prime $\land$ y is prime.

What does this mean? Try to read it: For every natural number greater than 2 denoted as n, if n is even, then we can find two other natural numbers, x and y so that $x + y = n$, or the sum of x and y is n, and x and y are prime numbers.

So, in simpler English, we can write every even number greater than 2 as the sum of two prime numbers. But now, we have a strategy, or plan to attack the problem now that it's an implication. We can start solving the problem, rather than asking "where do we start". So, for example, assume n is even and try to show x and y have the three properties.

This is called Goldbach's Conjecture. It's unsolved...

Answer 1:

First, we need to decipher what's being said. Then, we can solve the problem. For each of the subquestions, I'll write the english meaning so that someone in Grade 8 can (hopefully) understand it, and then answer it.

We'll put the implication truth table here.

x	y	$x \Rightarrow y$
T	T	T
T	F	F
F	T	T
F	F	T

1. Does there exist an integer x, so that $x^3 = 10$? Yes, we can choose $x = 30$.
2. Is every real number less than 0 or greater than 0? No, since 0 itself is neither.
3. If any real number x is -2, then is $x^2 = 4$? Yes. Remember, we only need to prove that if condition one is true, then condition two is also true. So it doesn't matter if condition one is false, or $x \neq -2$, even though this question asks for all real numbers.
4. If any real number $x^2 = 4$, then is it always true that $x = -2$? No, choosing $x = 2$ shows that $x^2 = 4$ but clearly, $x \neq -2$. Thus, our truth table would show that the implication is false. Remember, they're asking for every real number, so we need only one counterexample. If instead it was there exists an $x \in \mathbb{R}$ so that the above conditions hold, then it would be true.
5. If any real number $x^2 = 4$, then is it always true that $x = -2$ 0r $x = 2$?
6. Yes, since $\sqrt{4} = \pm 2$, so x is either -2 or 2 given $x^2 = 4$. Once again, refer to the truth table of implications if this doesn't make sense. If the first condition is false, then it doesn't matter what the second condition is.

   Now, as an extra question for you to ponder upon: Is the final question still true if it were an 'if and only if?'

Answer 2:

Like above, we'll write out the English version of each mathematical statement and proceed with proof or counterexample.

1. Does there exist a natural x so that we can find a natural y so that $x = -y$? This is false, since negative numbers aren't naturals.
2. For all real numbers x, is it true that every real y has the following property: If $x < y$, then $x^2 < y^2$? This is false! Take $x = -2$ and $y = 1$. Then, $x < y$ but $x^2 = 4 > 1 = y^2$.
3. For all naturals x, does there exist an integer y so that $x + y = 0$? This is true. Take $y = -x$. Thusm $y \in \mathbb{Z}$ and $x + y = 0$.
4. Does there exist an integer y so that every natural number denoted x has the property: $x + y = 0$? This is false. We can never find a y so that when we add it to every natural number, it always equals zero.

   Proof. Assume that the statement is true. Then, $y + 1 = 0 = y + 2$ as both 1 and 2 are natural numbers. But then we have that $1 = 2$. So we broke the laws of mathematics?!?! Thus, our assumption must have been false. □

There's two things to note here:

One, the order of quantifiers is important. This is the exact same as the true statement in the question above but just the order of quantifiers switched.

Two, our proof wasn't a traditional proof. It's called a proof by contradiction, or, we assume that the result is the opposite of what we want. Then, we break a preestablished law or rule. We'll talk about this more in our next video, if you haven't seen that one yet!

5. For every combination of 3 real numbers denoted x, y, z, we can find some other real denoted a so that $a + x = y + z$. This is true. Take $a = y + z - x$. Then, $a + x = y + z - x + x = y + z$. Since we only need to find one such a, we are done.

Answer 3:

This next question is a bit unorthodox, but they still include mathematical statements. So, let's deal with them one by one.

1. Is the following true: "Every vowel in the word Toronto is an o.". This is true. Check every vowel to see that yes, indeed, every vowel is an o.
2. Is the following true: "Every consonant in the word Toronto is a 't'.". This is false. r is a consonant in the word 'Toronto' but isn't a 't'.
3. Is the following true: "Every Greek letter in the word Toronto is a Σ."? This is true. Since there are no greek letters in 'Toronto', the implication truth table will always gives us true. This is an example of a vacuously true statement, which is true since we can't disprove yet. We have nothing to check.
4. Is the following true: "If x is a consonant in the word Toronto, then x is a 't'." This is the same as subquestion 2. Here, x denotes a letter. So, we can take $x =' r'$, and this is false as in subquestion 2.
5. Is the following true: "If x is a Greek letter in the word Toronto, then x is a Σ."? This is true. Like subquestion 3, there's nothing to check as 'Toronto' doesn't have any greek letters.

   On as aside: We formally call these objects (words or sequences of letters) strings, and each letter literally a letter. Doing proofs with objects like strings is very non-traditional but it comes up in CS quite a bit. And, the set of all strings is a set, so you can do math with them, such as set operations.
6. Is the following true: $(\forall x \in \mathbb{R})[x^2 < 0 \implies x > 0]$? Yes, this is vacuously true, since $x^2 \geq 0$, so the assumption, or the 'if' part or the antecedent is always false.

Answer 4:

This is super simple, but an easy way to understand 'if and only ifs'.

Proof. We prove the first implication, namely: if $x + y \geq x$, then $y \geq 0$. By cancellation, we have: $y \geq 0$.

Now we prove the second implication: If $y \geq 0$, then $x + y \geq x$. Then, add x on both sides to get: $x + y \geq y$.

Thus, the double sided implication has been proven. □