An Introduction to Mathematical Proofs

Implications

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When? Whenever You Watch

Recap

The last two videos, we covered axioms, mathematical statements, universal vs existential, conjunctions, negations and truth tables.

This is the third video out of four where we cover logic and how to prove things.

All this is very dense and hard to think about, and I doubt I did a good enough job at explaining it.

Don't worry, you can rewatch the videos and the worksheet is this time is very expansive and will get you comfortable with all these.

Here's An Old Proof

How would you prove this: if x is even, then x^2 is even?

All you would do is assume x is even, or assume the 'if' part and prove that x^2 is even, or prove the 'then' part.

This is called an implication, and there's actually a lot going on here. Let's examine this in detail by looking at the truth table.

Humble Beginnings

Let x, y be mathematical statements. We don't know whether they are true or false, so we'll write out all the possibilites in the table. What would 'if x then y' mean?

x	y
T	T
T	F
F	T
F	F

Implications

We write 'if x then y' as $x \Rightarrow y$, or x implies y. So let's slowly go through each case. What if x is true and y is true? Or, we can reach y knowing x is true?

x	y	$x \Rightarrow y$
T	T	T
T	F	
F	T	
F	F	

Implications

If x is true but y is false, then the whole implication must be false. Think about it: we can't reach y knowing x is true, so the statement is false.

An example: If x is prime, then $2x$ is prime. This is false.

x	y	$x \Rightarrow y$
T	T	T
T	F	F
F	T	
F	F	

Implications

This is where it's weird. If x is false, then it doesn't matter what y is, since we can never assume that x is true. So $x \Rightarrow y$ gives us no information, but we can never disprove this statement since x is never true.

This is called vacuous truth, or when something is true because it can't be false. For example, if lions can fly, then I need to drink water to live. Well, lions don't fly, so we can't disprove if I need to drink water or not.

x	y	$x \Rightarrow y$
T	T	T
T	F	F
F	T	T
F	F	T

Key Takeaway

If x is false, then the implication is true, no issues. So, all we need to show is that when x is true, y must also be true. Then, the implication is true.

We still have one more thing left to cover.

A Motivating Example

Consider this statement: If you eat a cookie, I will nuke Iceland. In our truth table, if you don't eat the cookie, I can still nuke Iceland and say I didn't lie (why lying matters at this point is beyond me).

But you object: It was a common understanding that the consequence of eating the cookie was that Iceland becomes Greenland 2.

But in fact, something deeper is happening. We also assume implicitly assume this: If you don't eat a cookie, then I won't nuke Iceland.

If And Only If

Consider this statement: If and only if you eat a cookie, I will nuke Iceland. Then, I can't nuke Iceland until you eat a cookie (as long as I abide by the truth).

This is not a simple implication. This is a double-sided implication, or an 'if and only if.'

In double sided implications, we show that $x \Rightarrow y$ and $\neg x \Rightarrow \neg y$. But shockingly, the second condition is the same as $y \Rightarrow x$. Think about this: If I nuke Iceland, then you ate a cookie, assuming the statement on this slide. This is the same as $y \Rightarrow x$. Only English is muddying up the logic.

An Example

$a \in \mathbb{Z}$ is even if and only if (we sometimes write iff) a is not odd.

How would you prove this? First, if a even, then a is not odd. This is the first implication: $x \Rightarrow y$.

Then, we have two methods. Either: if a is not odd, then a is even. So $y \Rightarrow x$, or if a is not even, then a is odd. This is $\neg x \Rightarrow \neg y$. And you can do both.

Interestingly, 'if and only if' acts like definitions. So we could replace x is even with x is not odd everywhere, and everything would be the same. So having multiple representations of one fact is useful.