Answers are found at the bottom of the worksheet on a seperate page.

Question 1: First, express $1+2+4+8+\cdots+2^n$ in sigma notation. Then, prove the following identity: $1+2+4+\cdots+2^n = 2^{n+1}-1$.

Question 2: Martine is playing a board game that uses meeple as a resource. She knows that:

A1. If she can win the game with n meeple, then she can also win with $n-3$ meeple (as long as $n-3 \geq 0$).

A2. If she can win the game with n meeple, then she can also win with $n-5$ meeple (as long as $n-5 \geq 0$).

A3. She knows how to win the game with 20 meeple.

For example, if $n = 3000$, then Martine can win with 2997 or 2995 meeple. In this case, we have no information on $n = 3000$.

What are all the amounts of meeple that Martine can win with? How does this change if (A3) is changed to (A3*): She knows how to win the game with 2020 meeple (instead of 20)?

Question 3: Show that $10^n - 1$ is divisible by 11 for every *even* natural number n.

Hint: We didn't talk about divisibilty yet. But here are three properties that will help you. Assume that a, b, c are naturals.

1: If a is divisible by b, this means we can write $a = bn$ for some $n \in \mathbb{N}$ and the reverse is true. Namely, if $a = bn$, then a is divisible by both b and n. For example, 21 is divisible by 7. Then, $n = 3$, or, $21 = 7 \cdot 3$. And, 21 is also divisible by 3.

2: If a is divisible by b, then for all c, ca is also divisible by b. This is true since multiplication doesn't affect divisibilty.

3: If a is divisible by b and c is divisible by b, then $a+c$ is divisible by b. This is true because $a = bn_1$ and $c = bn_2$ for suitable $n_1, n_2 \in \mathbb{N}$. Then, $a + c = bn_1 + bn_2 = b(n_1 + n_2)$ and by property 1, this is divisible by b.

If this doesn't make sense, play with some examples. If it still doesn't click, you can always skip the question or return later.

Question 4: Show that for large enough natural numbers n, we must have $n^3 < 3^n$.

Question 5: Usually, when we think of sequences, we list the first few elements and the pattern continues.

For example, $(a_n) = 1, 2, 4, 8, \cdots$ is a sequence denoted by (a_n). The first term in the sequence is $a_1 = 1$, the second term is $a_2 = 2$, the third term is $a_3 = 4$ and so on.

However, there's another way to define a sequence. A recursive definition. We define the first element and then apply a 'rule' for the later elements.

For example, here's a definition: Let $a_1 = 1$, and let $a_{n+1} = 3a_n + 1$ for $n \in \mathbb{N}$.

How would we compute a_2? Well, $a_2 = 3 \cdot a_1 + 1 = 3 \cdot 1 + 1 = 4$. Likewise, $a_3 = 3 \cdot a_2 + 1 = 13$.

Now, compute a_4, a_5, a_6 using this recursive definition. Then, prove that for all $n \in \mathbb{N}$ that $a_n = \frac{3^n-1}{2}$, or, the n^{th} term in our sequence can be found using a closed form.

Question 6: Show that every number can be written as a power of 2 multiplied by an odd number. For example, $56 = 2^3 \cdot 7, 13 = 2^0 \cdot 13, 20 = 2^2 \cdot 5$.

Question 7: Consider elements in $\mathbb{N} \times \mathbb{N}$. For example, $(1, 2) \in \mathbb{N} \times \mathbb{N}$. Now, we'll group elements together as follows: If $(a, b) \in \mathbb{N} \times \mathbb{N}$, then they will be in group $a + b$, denoted as G_{a+b}. So, $(1, 2) \in G_3$, or the group that sums up to 3. Likewise, $(7, 2) \in G_9, (11, 11) \in G_{22}, (1, 1) \in G_2$.

Now, we'll be investigating these groups. For example, $G_3 = \{(1, 2), (2, 1)\}$ while $G_4 = \{(1, 3), (2, 2), (3, 1)\}$. Find how elements are in G_n for all n and prove it.

Answer 1: Regarding sigma notation, $1 + 2 + \cdots + 2^n =$

$$\sum_{i=0}^{n} 2^n$$

Now, for the proof.

Proof. We'll prove this via induction. Our base case is $n = 1$. In this case, $1 + 2 = 3 = 4 - 1 = 2^2 - 1$. Thus, the base case holds. Proceed with the inductive step.

Assume that for some $k \in \mathbb{N}$, we have that $1 + 2 + \cdots + 2^k = 2^{k+1} - 1$. Now, we'll show that $1 + 2 + \cdots + 2^k + 2^{k+1} = 2^{k+2} - 1$.

We have:

$$\begin{aligned} &1 + 2 + \cdots + 2^k + 2^{k+1} \\ &= (2^{k+1} - 1) + 2^{k+1} \\ &= 2 \cdot 2^{k+1} - 1 \\ &= 2^{k+2} - 1 \end{aligned}$$

Thus, our proof is complete. So, $1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1$

□

Answer 2:

Proof. We observe that:

Starting meeple	**Result**	**Proof**
20	Win	20
19		
18		
17	Win	20-3
16		
15	Win	20-5
14	Win	20-3-3
13		
12	Win	20-5-3
11	Win	20-3-3-3
10	Win	20-5-5

Everything from 0 to 9 will be three less than a winning position, so by $A1$ she can win with all those as well.

If we use (A3*) instead of (A3), then she can win from all positions except $2020 - 1$, $2020 - 2$, $2020 - 4$, and $2020 - 7$.

□

Answer 3:

Proof. Our base case is $n = 2$. Then, $10^2 - 1 = 99 = 11 \cdot 9$. Thus, it's divisible by 11. Now, proceed with the inductive step.

Assume that $10^k - 1$ is divisible by some even $k \in \mathbb{N}$. Then, we want to show that the next even number, $k+2$, has the same property.

We have:

$$\begin{aligned}
&10^{k+2} - 1 \\
&= 100 \cdot 10^k - 1 \\
&= 100 \cdot (10^k - 1) + 100 - 1 && \text{We pull out 100 from: } 100 \cdot 10^k \\
&= 100 \cdot (10^k - 1) + 99
\end{aligned}$$

Since $(10^k - 1)$ is divisble by 11, so is $100 \cdot (10^k - 1)$. Likewise, since 99 is divisible by 11, so is the sum: $100 \cdot (10^k - 1) + 99$. Thus, this is divisible by 11 so our proof is complete. □

A cool note: The reason this only works for even numbers is because for odd numbers, we would get +9 at the end instead of +11. But interestingly, if we change the question to be divisible by 9, this holds for every natural.

Answer 4: While this question is on the tamer side, these types of questions require trial and error to find the correct base case.

Proof. First, we note that $3k^2 + 3k + 1 < k^3$ when $k \geq 4$. We'll prove this by induction. Let $k = 5$, we have: $48 + 12 + 1 < 64 = 4^3$.

Now for the inductive step, let $3k^2+3k+1 < k^3$ for some $k > 4$. We'll show that $3(k+1)^2+3(k+1)+1 < (k+1)^3$ Then,

$$\begin{aligned}
&3(k+1)^2 + 3(k+1) + 1 \\
&= 3k^2 + 6k + 3 + 3k + 3 + 1 \\
&= 3k^2 + 3k + 1 + 6k + 4 \\
&< k^3 + 6k + 4 \\
&= k^3 + 3k + 3 + 3k + 1 \\
&= k^3 + 3(k+1) + 3k + 1 \\
&< k^3 + 3k^2 + 3k + 1 \\
&= (k+1)^3
\end{aligned}$$

Now, continue with our main proof. Our base case is $n = 4$. In this case, we have: $4^3 = 64 < 81 = 3^4$. Now, proceed with the inductive step.

Let $k^3 < 3^k$ for some $k > 4$. Then, we'll show that $(k+1)^3 < 3^(k+1)$.

We have:

$$\begin{aligned}
&(k+1)^3 \\
&= k^3 + 3k^2 + 3k + 1 \\
&< k^3 + k^3 \\
&< 3 \cdot k^3 \\
&< 3 \cdot 3^k \\
&= 3^{k+1}
\end{aligned}$$

Thus, our proof is complete.

□

Answer 5:

We'll list out a_4, a_5, a_6.

$a_4 = 3 \cdot a_3 + 1 = 3 \cdot 13 + 1 = 40$

$a_5 = 3 \cdot a_4 + 1 = 3 \cdot 40 + 1 = 121$

$a_6 = 3 \cdot a_5 + 1 = 3 \cdot 121 + 1 = 364$

Now, we'll proceed with the proof. This proof only includes necessary details. Just to switch it up a bit.

Proof. Note that $\frac{3^1-1}{2} = \frac{2}{2} = 1 = a_1$.

Now, suppose that $a_n = \frac{3^n-1}{2}$ for a particular $n \in \mathbb{N}$.

Note that

$$\begin{aligned} a_{n+1} &= 3a_n + 1 && \text{By definition} \\ &= 3\frac{3^n - 1}{2} + 1 && \text{By IH} \\ &= \frac{3 \cdot 3^n - 3 + 2}{2} \\ &= \frac{3^{n+1} - 1}{2} \end{aligned}$$

As desired.

□

Note: We never explicitly state the induction hypothesis. While this is nitpicking, our induction hypothesis is: ''$a_n = \frac{3^n-1}{2}$ for some $n \in \mathbb{N}$'' and not just ''$\frac{3^n-1}{2}$''. The latter isn't a mathematical statement, it's just an equation. It has no meaning without being compared to something else. Of course, this doesn't change your proof, but it's always good to be careful about things and what our induction hypothesis formally is.

Answer 6:

Proof. Let $P(n)$ be the statement "n can be written as a product of a power of 2 and an odd number." We proceed by strong induction.

$\boxed{\text{Base}}$. If $n = 1$, then $1 = 2^0(1)$, and 1 is odd.

$\boxed{\text{Induction}}$. Suppose $P(1), P(2), \ldots, P(n)$ are true for a particular n.

<u>Case 1</u>. If $n + 1$ is odd, then $n + 1 = 2^0(n + 1)$ is the desired result.

<u>Case 2</u>. If $n + 1$ is even, then there is a (positive) integer m with $2m = n + 1$. Since $P(m)$ is true, there are integers a and odd b such that $m = 2^a b$. Therefore

$$n + 1 = 2m = 2 \cdot 2^a b = 2^{a+1} b,$$

as desired. □

Answer 7:

G_n has $n - 1$ many elements. We'll prove this by induction.

Proof. Our base case is $n = 2$. We see that $G_2 = \{(1, 1)\}$ has only one element. Regarding $n = 1$, it has no elements. Now, the inductive step.

Assume that G_k has $k - 1$ elements for some $k \in \mathbb{N}$. We'll show that G_{k+1} has k elements.

List all the elements in G_k:

$$G_k = \{(1, k-1), (2, k-2), \ldots, (k-2, 2), (k-1, 1)\}$$

Now, add 1 to the first coordinate in each element in G_k. We get:

$$\{(2, k-1), (3, k-2), \ldots, (k-1, 2), (k, 1)\}$$

All of these elements are in G_{k+1} by definition and there are $k - 1$ of them. However, we fail to list $(1, k)$ as $(0, k) \notin \mathbb{N} \times \mathbb{N}$.

This is the only element we fail to list as every natural between 1 and $k+1$ is present in the first coordinate when we take elements from G_k to G_{k+1}.

We see that $(1, k) \in G_{k+1}$, so there are exactly $k - 1 + 1 = k$ elements in G_{k+1}. □

We could also prove this directly by looking at sums of natural numbers.

Proof. Let $n \in \mathbb{N}$. This problem is equivalent to finding how many pairs of a and b sum to n. Since $a, b \neq 0$, we can never have $0 + n$ as a solution.

Then, we can add 1 to a and subtract 1 from b to get a new solution.

Since there are exactly $n - 1$ naturals between 0 and n, we can repeat this process at most $n - 1$ times, so there must be $n - 1$ ways to sum naturals to n. Thus, G_n has n many elements. □