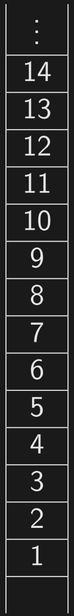
An Introduction to Mathematical Proofs

Induction

Who? Fahad Hossaini

When? Whenever You Watch

A Ladder



First, Start At The Bottom

⋮
14
13
12
11
10
9
8
7
6
5
4
3
2
1

Then, Go Up!

⋮
14
13
12
11
10
9
8
7
6
5
4
3
2
1

Then, Go Up!

⋮
14
13
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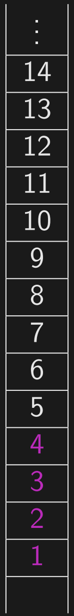
Then, Go Up!



Then, Go Up!

$\vdots$
14
13
12
11
10
9
8
7
6
5
4
3
2
1

We Want To Show This:

First, we can reach step 1, or show our closed form works for 1.

Then, if we can reach step k, then we can also reach step $k + 1$.

Mathematically, if we think about closed forms, prove the formula for step $k + 1$ knowing that the formula works for step k.

Staircase?

							$\iddots$
						7	
					6		
				5			
			4				
		3					
	2						
1							

Say Hi To Benny

ribbit • *ribbit*

Yes, my drawing skills are subpar, so Benny is a dot.

1 2 3 4 5

The Plan Is Simple! Start!

The Plan Is Simple! Jump!

The Plan Is Simple! Jump!

The Plan Is Simple! Jump!

The Plan Is Simple! Jump!

1 2 3 4 •

The Plan Is Simple! Jump!

Now, Benny's Turn!

With Benny's help, we'll prove the following formula:

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

You'll notice we use n here and not k. We'll use n when we talk about the formula itself and k when we're talking about a specific number.

So, let's prove this via the Principle of Mathematical Induction!

Base Case!

Proof. Our first step is showing Benny can reach the first lilypad, or that our formula holds for $k = 1$.

Let's confirm this! The left hand side / summation will just be 1 whereas our right hand side / formula is $\frac{1\cdot(1+1)}{2} = 1$.

Since both sides are equal, our formula holds for $k = 1$, or, our first lilypad/step is reachable. $\square$

Induction Hypothesis

Proof. Continued.

Now, we'll show that from any arbitrary lilypad, Benny can jump to the next one.

The assumption that Benny can reach some arbitrary lilypad is called our Induction Hypothesis.

Regarding our closed form, this is assuming that the formula holds for some $k \in \mathbb{N}$. Then, we'll show that the formula holds for $k + 1$.

Remember our diagrams? 1 then 2 then 3 and so on. But we can't explcitly show this for every number, so we assume the most general form, any random or arbitrary number.

Inductive Step

Proof. Continued.

Showing that our formula or claim works for $k+1$ using our induction hypothesis is called the inductive step.

We want to show that:

$$1+2+\cdots+k+(k+1)=\frac{(k+1)(k+2)}{2}$$

It's crucial you understand why this is what we want to prove.

Now, let's solve $1+2+\cdots+k+(k+1)$! We can use our induction hypothesis directly here and substitute $1+2+\cdots+k$ with $\frac{k(k+1)}{2}$. We get:

$$\frac{k(k+1)}{2}+(k+1)$$

Completing Our Inductive Step

Proof. Continued.

Now, all we need to do is some simple algebra now that the steps are clear. We get:

$$\begin{aligned} 1 + 2 + \cdots + k + (k+1) &= \frac{k(k+1)}{2} + k + 1 \\ &= \frac{k^2+k}{2} + \frac{2k+2}{2} \\ &= \frac{k^2+3k+2}{2} \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

We've now shown that our formula holds for $k+1$ assuming that it holds for some arbitrary k.

Proof Done!

Proof. Continued.

We've now shown that our formula holds for $k+1$ assuming that it holds for some arbitrary k. In combination with our base case, the formula must hold for every natural number. Thus, we have shown that:

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

$\square$

Let's Do One More! But A Little Faster This Time.

Prove that for all $n \in \mathbb{N}$:

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof. We start with our base case, the first lilypad.

The LHS/sum is 1 while the RHS/formula is: $\frac{1(1+1)(2+1)}{6} = \frac{6}{6} = 1$.

$\square$

Inductive Hypothesis

Proof. Continued.

Assume that our formula works for some $k \in \mathbb{N}$. We want to show that our formula works for $k + 1$.

Or in frog terms, assume that Benny can reach lilypad k. Show that Benny can jump to lilypad $k + 1$.

We want to show that:

$$1^2 + 2^2 + \cdots + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

$\square$

Inductive Step

Proof. Continued.

We have the following:

$$
\begin{aligned}
& 1^2 + 2^2 + \cdots + k^2 + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{(k^2+k)(2k+1)}{6} + (k^2+2k+1) \\
&= \frac{2k^3 + k^2 + 2k^2 + k + 6k^2 + 12k + 6}{6} \\
&= \frac{2k^3 + 9k^2 + 13k + 6}{6}
\end{aligned}
$$

Let's Handle The RHS

Proof. Continued. We'll expand: $\frac{(k+1)(k+2)(2k+3)}{6}$ and get:

$$
\begin{aligned}
&\frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{(k^2+k)(2k+3)}{6} \\
&= \frac{2k^3+3k^2+6k^2+9k+4k+6}{6} \\
&= \frac{2k^3+9k^2+13k+6}{6}
\end{aligned}
$$

Since both the LHS and RHS are equal, the inductive step is complete so our formula holds for all natural numbers. □

Final Tip

Use your induction hypothesis!

If you don't use your induction hypothesis, you'll get stuck somewhere! It's usually important that Benny is on the previous lilypad in order to jump to the next one.

If you prove a proof without needing the induction hypothesis, you either made a mistake or induction is overkill.