Answers are found at the bottom of the worksheet on a seperate page.

Note: It's okay if you watch the next video and do that worksheet before this one, since this one is harder in my opinion. But you are free to do whatever.

Here, we have a few proof questions to get you used to proving things. Q1 - Q3 are the hardest in my opinion.

Question 1: Show that if $0 < a < b$ and $0 < c < d$, then $a \cdot c < b \cdot d$.

Question 2: What would change in the above proof if instead of $0 < c < d$, we had: $c < d < 0$?

Question 3: If $a > 1$, show that $\sqrt{a} < a$. What happens if $a = 1$ or $a < 1$?

Question 4: Show that $x^4 \geq 0$. Can you generalize this for all positive even powers?

Question 5: Show that $a^3 \leq |a|^3$. Can you generalize this for all odd positive powers?

Question 6: Show that $|ab^2| = |a| \cdot b^2$.

Question 7: Use the triangle inequality to find a number M such that if $-2 \leq x \leq 0$:

$$|-x^3 - 3x^2 + 6x + 1| \leq M$$

It is known (e.g. through calculus) that the smallest M that works is 15. Explain why your M is larger than 15.

Question 8: This is called the Reverse Triangle Inequality: $|a - b| \geq ||a| - |b||$. Show that the Reverse Triangle Inequality is true. You can use the regular Triangle Inequality, but it's unnecessary.

Question 9: This is called the AGM or AMGM Inequality:

$$xy \leq \left(\frac{x+y}{2}\right)^2$$

.

In the case that $xy > 0$, we can square root both sides to get:

$$\sqrt{xy} \leq \frac{x+y}{2}$$

Now, show that $4xy \leq (x+y)^2$ using the first inequality

Question 10: Now an implementation of the AGM Inequality:

What's the maximum area a square can have if the perimeter has to be 24? Prove your answer using the AGM Inequality.

Question 11: Prove the AGM Inequality. Hint: The proof will start with the following fact: $(x-y)^2 \geq 0$. Or, in other words, start with the AGM Inequality and reach this statement.

Question 12: Use AMGM to prove that for all non-negative x values:

$$[x(x+2) \leq (x+1)^2]$$

Question 13: Use AMGM (3 times!) to prove: $\forall x, y, z, w \geq 0$

$$\frac{x+y+z+w}{4} \geq \sqrt[4]{xyzw}$$

Remark: In both the Triangle Inequality and the AGM Inequality, we don't have strict inequalities. We can get cases when both sides are equal. Can you figure out when we can get equality for both?

Answer 1:

Proof. We'll follow a very similar proof as found in the video.

Start with $a < b$. Multiply by c on both sides: $a \cdot c < b \cdot c$.

We also have: $c < d$, so we can multiply both sides by b to get: $c \cdot b < b \cdot d$.

Combining these, we get: $a \cdot c < b \cdot c < b \cdot d$.

By rule 2, we have that: $a \cdot c < b \cdot d$

□

Answer 2: Letting c and d be negative would flip the inequality. So we would have the following: If $0 < a < b$ and $c < d < 0$, then $a \cdot c > b \cdot d$. Let's prove this.

Proof. We'll follow a very similar proof as in Answer 1.

Start with $a < b$. Multiply by c (negative) on both sides: $a \cdot c > b \cdot c$.

We also have: $c > d$, so we can multiply both sides by b (positive) to get: $c \cdot b > b \cdot d$.

Combining these, we get: $a \cdot c > b \cdot c > b \cdot d$.

By rule 2, we have that: $a \cdot c > b \cdot d$

□

Answer 3:

Proof. Since $a > 1$, we multiply by a on both sides to get: $a^2 > a$.

Now, we apply rule 4 to get: $a^2 > \sqrt{a}^2$.

We square root both sides and get that: $a > \sqrt{a}$.

□

But why are we allowed to square root both sides? Remember, this is the same as showing: if $0 < a < b$, then $\sqrt{a} < \sqrt{b}$. We'll show this really quickly (this one was harder than I first thought, so I'll redo it here).

Proof. We have: $a < b$ which is the same as $0 < b - a$.

We now use the difference of squares formula: $x^2 - y^2 = (x + y)(x - y)$.

In our case, $x = \sqrt{a}$ and $y = sqrtb$. So, we get that: $a - b = (\sqrt{b} - \sqrt{a})(\sqrt{b} + \sqrt{a}) < 0$.

Now, we multiply both sides by $\frac{1}{(\sqrt{b}+\sqrt{a})}$. Note that this number is positive as the denominator is positive as the square roots are positive. Then we have:

$\sqrt{b} - \sqrt{a} < 0$. Rearrange to get:

$\sqrt{a} < \sqrt{b}$

□

If $a = 1$, we get equality. If $a < 1$, then $\sqrt{a} > a$.

Answer 4:

Proof. Since $x^4 = x^2 \cdot x^2$ and $x^2 > 0$, we have that: $x^2 \cdot x^2 > 0 \cdot 0$. Thus, $x^4 > 0$. □

For any even number y, we can break x^y into $\frac{y}{2}$ many pieces of x^2 and apply the same process as above.

Answer 5:

Proof. We have: $a^3 = a^2 \cdot a \leq |a|^2 \cdot |a| = |a|^3$. □

For any positive odd power x, we can apply Answer 4 to the even power, and always be left over with a. Then, we apply $a < |a|$ and can get that $a^x \leq |a|^x$.

Answer 6:

Proof. We have: $|ab^2| = |a||b^2| = |a| \cdot b^2$. □

Answer 7:

Proof.

$$\begin{aligned} |-x^3 - 3x^2 + 6x + 1| &\leq |-x^3| + |3x^2| + |-6x| + |1| \\ &\leq |x^3| + 3|x^2| + 6|x| + 1 \\ &\leq |x|^3 + 3|x|^2 + 6|x| + 1 \\ &\leq 2^3 + 3 \cdot 2^2 + 6 \cdot 2 + 1 = 33 \end{aligned}$$

Note that the largest possible value of $|x|, |x|^2$ and $|x|^3$ is when $x = -2$. The reason this value is less than 15 is because we are ignoring all possible cancellation of the terms by using the triangle inequality. The number 15 takes the cancellation into account. □

Answer 8: We first start with reversing the steps to see what true fact we can start with and derive the Reverse Triangle Inequality.

We have: $|a - b| \geq ||a| - |b||$. I'll show only the algebraic steps.

$$\begin{aligned} |a - b| &\geq ||a| - |b|| \\ |a - b|^2 &\geq ||a| - |b||^2 \\ |a - b|^2 &\geq |a|^2 - 2|a||b| + |b|^2 \\ (a - b)^2 &\geq a^2 - 2|a||b| + b^2 \\ a^2 - 2ab - b^2 &\geq a^2 - 2|a||b| + b^2 - ab > -|a||b|ab < |a||b| \end{aligned}$$

Note: In the regular Triangle Inequality, we have that: $|x| + |y| = ||x| + |y||$. But this is not the case here, so we add the outer absolute values on the right hand side to get a stricter inequality (since $||x| - |y|| \geq |x| - |y|$).

Now, we present the formal proof.

Proof. Here's the complete proof:

We know that $xy \leq |xy|$ by rule 1 of absolute values. Then, we multiply by negative two and expand the right hand side by rule 4 to get:

$$-2xy \geq -2|x||y|$$

Now, we add x^2 and y^2 to both sides. Then, we note that since $x^2 = |x|^2$ and $y^2 = |y|^2$ by rule 2, we have the following:

$$x^2 - 2xy + y^2 \geq |x|^2 - 2|x||y| + |y|^2$$

Now, we use the algebraic identity: $(x-y)^2 = x^2 - 2xy + y^2$ on both sides to get:

$$(x-y)^2 \geq (|x| - |y|)^2$$

Finally, after square rooting both sides, we can apply rule 3 to the left side to get:

$$|x-y| \geq ||x| - |y||$$

□

Answer 9:

I'll show only the algebraic steps:

$$\begin{aligned} xy &\leq (x + y/2)^2 \\ xy &\leq (x+y)^2/4 \\ 4xy &\leq (x+y)^2 \end{aligned}$$

Answer 10:

Proof. We start by noting that the area of a square is given by: $A = lw$ and the perimeter is given by $P = 2l + 2w$.

In our case, the perimeter is 24, so we have that $l + w = 12$.

Then, by AGM, $A = lw \leq \left(\frac{12}{2}\right)^2 = 36$.

Thus, the maximum area of the square is 36 units squared. □

Answer 11: The rough work will be omitted and we'll jump directly into the proof.

Proof. Start by noting that $(x-y)^2 \geq 0$.

Then, expand the LHS to get: $x^2 - 2xy + y^2 \geq 0$.

Add $4xy$ to both sides to get: $x^2 + 2xy + y^2 \geq 4xy$.

Now, apply the algebraic identity to the LHS: $(x+y)^2 \geq 4xy$.

Finally, flip the inequality direction, divide by 4 on both sides, write $4 = 2^2$ and factor the square to get the AGM Inequality:

$$xy \leq \left(\frac{x+y}{2}\right)^2$$

□

Answer 12:

Proof. Note that $x+1$ is the average (arithmetic mean) of x and $x+2$. Also, both terms are non-negative. Therefore:

$$\sqrt{x(x+2)} \leq \frac{x + (x+2)}{2} = x + 1$$

Squaring both sides gives (since both sides are non-negative):

$$x(x+2) \leq (x+1)^2$$

□

Answer 13:

Proof. Applying AMGM to x, y gives:

$$\frac{x+y}{2} \geq \sqrt{xy}.$$

Applying AMGM to z, w gives:

$$\frac{z+w}{2} \geq \sqrt{zw}.$$

Applying AMGM to $\frac{x+y}{2}$ and $\frac{z+w}{2}$ gives:

$$\frac{\frac{x+y}{2} + \frac{z+w}{2}}{2} \geq \sqrt{\frac{x+y}{2}\frac{z+y}{2}} \geq \sqrt{\sqrt{xy}\sqrt{xy}} = \sqrt[4]{xyzw}.$$

□