An Introduction to Mathematical Proofs

Logic

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When? Whenever You Watch

Axioms

Axioms are rules we take to be true without proofs.

We come up with axioms to formalize the most basic things, similar to defintions.

But remember, axioms are rules. For example, in chess, the axioms are the rules of the game, how piece move are the definitions and tactics and openings are the actual theorems.

And it gets confusing because how the pieces move ARE parts of the rules of the game. But axioms are the rules and are unchangable, like how to win the game but we can add definitions, like new pieces and how they move.

All This To Say...

Axioms are the only things we can work with. They're usually simple, and give us a foundation to work from.

We won't delve into why axioms are chosen, since that's an even more philosophical question.

We start with axioms and prove theorems and statements, and keep doing so. Understanding this helps us understand why we do proofs and what we can work with.

Mathematical Statements

Definition Mathematical Statement: A statement that is either true or false.

An opinion or question is not a statement. Also, mathematical statements don't have to be mathematical in nature. We just call it that since theorems are precisely mathematical statements that we prove from axioms or previous theorems and statements.

Example

Example

"The sky is always blue."

This is a mathematical statement. But it isn't true. Let's say we want to change it to be true, what can we alter?

Example But Better

Example

"The sky is always blue from 2-3pm in Iceland."

This is still a mathematical statement. But it's false!

Worse Than Example But Better But Better Than Example

Example

"The sky has been blue at a point in time in the past."
Still mathematical statement, and finally true.

Another Example Slightly Worse Than In Worse Than Example But Better But Better Than Example

Example

6 is prime.

Still mathematical statement. But false.

Okay, Enough With The Super Long Truly Terrible Tongue-Twisty Titles

Example

I like blue.

Not a mathematical statement. Reason: Opinion, not a statement.

Quantifiers

Definition Universal Quantifiers: A mathematical statement made about all or every of object x.

Example Every even number added with an even number is even.

Definition Existential Quantifiers: A mathematical statement made about a single version of object x.

Example Find a solution to $x^2 + 3x - 2 = 0$.

Structure

Having words to describe these things give us a structure and surefire method to tackle these problems. We can figure out where the starting line is, or what the general objective is.

In this case, we are using logic and english as a tool to solve mathematics.

While some of this stuff is too much information at our level, it provides a really good starting point to understand proofs at the simplest level.

A Few More Mathematical Statements

Here, decide if a statement is universal (for all) or existential (there exists). Also, see if you can tell if they're true or false.

Example 1. Every prime number is odd.

Example 2. There exists a rectangular prism with integers values for length, width, height, all three face diagonals. This is also called a Euler brick or an Euler cuboid.

Example 3. There exists non-zero valued integer solutions for the following equation: $a^3 + b^3 = c^3$.

Symbols

$\forall$ means "For all" or "For every". It acts as the universal quantifer. Simplest example: $\forall x \in \mathbb{R}$.

So saying: Every integer is prime. is the same as
$\forall x \in \mathbb{Z}$, x is prime.
$\exists$ means "There exists" or "For some". It acts as the existential quantifer. Simplest example: $\exists y \in \mathbb{Q}$.

So saying: There exists an integer that is prime, or there's a prime integer. is the same as: $\exists x \in \mathbb{Z}$, x is prime.

Translating from English to symbols is hard! As seen in the second example, there are two ways to write out the mathematical statement, and one is easier to translate into symbols than the other.

Order Of Quantifiers

$\forall x, \exists y$ is *very* different than saying $\exists x, \forall y$.

Take the following example: x is a factor of y and $x \neq 1, y$.

Let's add quantifers: $\forall x \geq 2 \in \mathbb{Z}, \exists y \in \mathbb{Z}$ such that x is a factor of y and $x \neq 1, y$.

Now the other way: $\exists x \geq 2 \in \mathbb{Z}, \forall y \in \mathbb{Z}$ such that x is a factor of y and $x \neq 1, y$.

The first one says: For all integers x, we can find some integer y so that x is a factor of y and x is not equal to 1 or y.

The second one says: There exists integer x so that for all integers y, x is a factor of y and x is not equal to 1 or y.

See The Issue?

The first one says: For all x, we can find some y so that x is a factor of y and x is not equal to 1 or y.

The second one says: There exists x so that for all y, x is a factor of y and x is not equal to 1 or y.

The first one is true but the second one is false! The first one says that every integer greater than 2 is a factor in another integer. But the second one says that we can find an integer greater than 2 that is a factor in EVERY integer! Vastly different!

Recap!

Axioms are rules we take for granted, or rules that don't need to be proved.

A mathematical statement is a statement that is true or false. These are what we want to prove, and what we call theorems.

$\forall$ means "For all" and is a universal quantifer. It talks about every instance or version of something.

$\exists$ means "There exists" and is an existential quantifer. It talks about one instance or version of something,