An Introduction to Mathematical Proofs

Numbers

Who? Fahad Hossaini

When? Whenever You Watch

The Simplest Numbers

The simplest numbers are the counting numbers. We call them the natural numbers. Or, all positive non-decimal numbers.
We use the symbol $\mathbb{N}$ to denote this.

Definition

$$\mathbb{N} = \{1, 2, 3, 4, 5, 6, \ldots\}$$

Now, we add in some negatives and 0 so we can subtract, and we get the integers, or the set of all non-decimal numbers. We'll call them integers from this point on.

Definition

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

The Next Number Systems

Definition

In order to divide, we need to introduce ratios, or fractions. This number system is called the rationals.

$$\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z} \text{ and } b \neq 0\}$$

There's something we have to mention: We only consider fractions in their simplest form when talking about elements in $\mathbb{Q}$. So, $\frac{a}{b}$ is in their lowest or simplest form (smallest denominator).

But what about π or $\sqrt{2}$? How do we add them in? Well, we literally add them in and call this the set of all numbers, the real numbers.

We use $\mathbb{R}$ to denote the set of all real numbers, or any number we can think of, even values that can't be represented as a fraction.

Eventually Periodic

Write every number in their decimal form. Then, all rationals eventually repeat. For example, $\frac{13}{3} = 4.333\ldots$. But numbers in $\mathbb{R}$ don't need to repeat. So, maybe you can see how many numbers are in $\mathbb{R}$ but not in $\mathbb{Q}$.

Prove That $\sqrt{2}$ is Irrational

Proof. Assume for the sake of contradiction that $\sqrt{2}$ is irrational. Then, we can write $\sqrt{2} = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$ where $\frac{a}{b}$ is in their simplest form.

After squaring both sides, we get the following:
$2 = \frac{a^2}{b^2}$

Then, multiply by b^2 on both sides.
$2b^2 = a^2$

So, a^2 is even. Thus, a is even. Write a as $2k$ to get:
$2b^2 = (2k)^2 = 4k^2$

Finally, divide by two on both sides to get:
$b^2 = 2k^2$

Here's The Kicker

Proof. Continued.
But wait, then b^2 is even so b is even. Thus, $\frac{a}{b}$ couldn't have been in simplest form? How? Well, it must be that $\sqrt{2}$ can't be represented as a rational. So, $\sqrt{2}$ must be irrational. $\square$

Wow. It's beautiful, but tough. Since this was a short one, I had to challenge you a bit in a different way. While understanding the proof isn't pivotal, if you do, you'll be able to tackle everything from this coruse.