An Introduction to Mathematical Proofs

Sets

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When? Whenever You Watch

Introduction to Sets

Definition

Set: An unordered collection of unique objects.

Example

$A = \{1, 2, 3\}$
$B = \{\text{cat,dog,buffalo}\}$

More Examples

Example

$C = \{2, 5, 7\}$
$D = \{10, 17, 10\}$. D isn't a set.
$A = \{3.5, 8, 1, 6, -3\}$

Why are Sets Useful?

Abstraction: An unordered collection of unique objects can essentially describe any group of collection of things we may want to study mathematically.

Formality: We have formal definitions to work off of. Now, we know what it means to have a 'collection' of objects.

So now, when we try to define numbers, we represent a number as an element from a collection of many numbers. Trust me, it helps.

Additionally, simple defintions use sets. For example, formally defining functions requires sets. We need formalization for rigor.

How Many More Examples Of Sets?

Short answer: Many more.

Example $A = (0, 1)$. This is the open interval from 0 to 1. So, $0.5 \in A$ but $1 \notin A$.
$D = [0, 1]$. This is the closed interval from 0 to 1. Now, $1 \in D$.

Now, we'll use Set Builder notation.

Example $B = \{x : x > 3 \text{ and } x \text{ is even}\}$. So, $4 \in B$ while $2 \notin B$ and $5 \notin B$.

Sets in Sets

Also, we can have sets be elements in other sets. For example, if $A = \{1, 2\}$, it's perfectly fine for $B = \{A, 4, 5\}$.

Here are a few things to note: $1 \notin B$. B only has three elements, they are the set $A, 4$ and 5. So, $\{1, 2\} \in B$. We need to be careful here.

Subsets

But how do we talk about relationships between sets? Here's a good example. Let $A = \{1, 2\}$ and $B = \{1, 2, 3\}$. Since all the elements in A are in B, we say that A is a subset of B, or, $A \subseteq B$.

So, what if $A = \{1, 2, 4\}$? Then, A is not a subset of B, or, $A \not\subseteq B$.

Subsets are an important concept because they indicate some sort of relationship between sets. But here's also something just as important: What if two sets are equal? i.e. what if we have two different representations for the same set. HOW do we prove they are equal?

We'll answer this soon.

What Can we do With Sets?

Let $C = \{1, 5, 9\}$ and $D = \{5, 6, 8\}$. What operations might we want to easily work with these sets?
We'll order the sets for easy readability, but it's not necessary.

Intersection

Let A and B be any two sets. For our example, let $C = \{1, 5, 9\}$ and $D = \{5, 6, 8\}$.

Definition $A \cap B = \{x : x \in A \text{ and } x \in B\}$

Example $C \cap D = \{5\}$

Intersection of two sets produces the elements they share.

Union

Let A and B be any two sets. For our example, let $C = \{1, 5, 9\}$ and $D = \{5, 6, 8\}$.

Definition $A \cup B = \{x : x \in A \text{ or } x \in B\}$

Example $C \cup D = \{1, 5, 6, 8, 9\}$

Union of two sets produces the combined elements.

Difference

Let A and B be any two sets. For our example, let $C = \{1, 5, 9\}$ and $D = \{5, 6, 8\}$.

Definition $A \setminus B = \{x : x \in A \text{ and } x \notin B\}$

Example $C \setminus D = \{1, 9\}$

Difference of two sets produces the elements in one set that isn't in the other.

Complement

Let A be a set. Let U be our universal set, or all the largest set we want to work with. In the context of numbers, these will be the set of every number. For our example, we'll let U be all positive non-decimal numbers and let $D = \{5, 6, 8\}$.

Definition $A^C = \{x : x \in U \text{ and } x \notin A\}$

Alternatively; $A^C = U \setminus A$

Example $D^C = \{1, 2, 3, 4, 7, 9, 10, 11, \ldots\}$

Complement of a set produces all the elements NOT in the set.

Cartesian Product

Let A and B be any two sets. For our example, let $C = \{1, 5, 9\}$ and $D = \{5, 6, 8\}$.

Definition $A \times B = \{(x, y) : x \in A \text{ and } y \in B\}$

Here, we produce pairs of elements as seen in Set Builder, not just an element by itself, or a number.

Example $C \times D = \{(1, 5), (1, 6), (1, 8), (5, 5), (5, 6), (5, 8), (9, 5), (9, 6), (9, 8)\}$

Cartesian Product of two sets produces all pairs of elements, where the first index comes from A and the second index comes from B.

Set Equality

Now we'll answer the question about set equality now that we explain the process and do a proof. We say two sets are equal when they contain the same elements. Thus, they represent the same unordered collection of unique objects.

Take, for example,
$A = \{1, 3, 5, 7, \dots\}$, $B = \{2, 4, 6, 8, \dots\}$ and
$\mathbb{N} = \{1, 2, 3, 4, 5, \dots\}$. Prove that $\mathbb{N} = A \cup B$.

The idea is this: If we can show that $A \cup B$ is a subset of $\mathbb{N}$, then we showed that every element in $A \cup B$ is in $\mathbb{N}$.

Conversely, if we can also show that $\mathbb{N} \subseteq A \cup B$, then we showed that every element in $\mathbb{N}$ is in $A \cup B$.

Set Equality Continued

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Now, if both are true, then every element in $\mathbb{N}$ is in $A \cup B$ and vice versa, so they have the same elements. We call this proof a double subset inclusion and this proves that the sets are equal.

Proof!

Proof.

We start by showing that $\mathbb{N} \subseteq A \cup B$.

Take an arbitrary element $n \in \mathbb{N}$. If n is odd, then $n \in A$. Otherwise, $n \in B$. Since n is in either set, we must have that $n \in A \cup B$ by definition.

Thus, $\mathbb{N} \subseteq A \cup B$. Now, we'll show the other direction: $A \cup B \subseteq \mathbb{N}$.

Take an arbitrary element $x \in A$. Then, x is positive and clearly a non-decimal number since it is odd. Thus, $x \in \mathbb{N}$. A similar process shows that if $x \in B$, then x is a non-decimal positive number, so $x \in \mathbb{N}$.

So, $A \cup B \subseteq \mathbb{N}$. Now that both subset inclusions have been proven, by definition, we have that $A \cup B = \mathbb{N}$.

Null Set

$\varnothing$ is the symbol we use to denote the null set, a set with no elements.

For example, how would we say that a set contains nothing? We would say $A = \varnothing$ to denote that A is an empty set.

We say that every set is a superset of $\varnothing$, or, $\varnothing \subseteq A$ for all sets A.

While the null set doesn't come up often, we must always consider it when making a claim about sets. Sometimes, the null set is the only counterexample to a claim, so we need to be careful regarding this.

The Most Unintuitive Part Of This Lecture

But this is where it gets bad: If $A = \{\varnothing\}$ and $B = \{A, 4, 5\}$, then both $\varnothing \subseteq B$ and $\{\varnothing\} \in B$. Here, A is a set that contains the null set as an element. That's why this gets confusing quick.

Cheatsheet: Basic Set Knowledge

Set: An unordered collection of unique objects

Bracket Types:

$\{\}$ - Listing of elements, finite.

Example $A = \{0, 1\}, 0.5 \notin A, 1 \in A$

$()$ - Open interval, do NOT include endpoint.

Example $A = (0, 1), 0.5 \in A, 1 \notin A$

$[]$ - Closed interval, include endpoint.

Example $A = [0, 1], 0.5 \in A, 1 \in A$

Set Builder Notation: $A = \{form : condition\}$

Example $A = \{x : x \geq 3 \text{ and } x \text{ is even}\}$, $4 \in A, 2 \notin A$

Cheatsheet: Set Operations

Let $C = \{2, 4, 7\}$, $D = \{5, 7, 8\}$ and U be the set of all positive non-decimal numbers.

Example Intersection: $C \cap D = \{7\}$

Example Union: $C \cup D = \{2, 4, 5, 7, 8\}$

Example Difference: $C \setminus D = \{2, 4\}$

Example Complement: $D^C = \{1, 2, 3, 4, 6, 9, 10, 11, \ldots\}$

Example Cartesian Product: $C \times D = \{(2, 5), (2, 7), (2, 8), (4, 5), (4, 7), (4, 8), (7, 5), (7, 7), (7, 8)\}$